ML2R Coding Nuggets
Intersection String Kernels for Language Processing

Christian Bauckhage
Machine Learning Rhine-Ruhr
Fraunhofer IAIS
St. Augustin, Germany

ABSTRACT

This is the first in a miniseries of notes on kernel methods for language processing. We discuss the idea of measuring *n*-gram similarities of words by computing intersection string kernels and demonstrate that the *Python* standard library allows for compact implementations of this idea.

1 INTRODUCTION

Most machine learning methods for natural language processing require numerical vector representations of words. Nowadays, such **word embeddings** are usually computed using context-free or bi-directional contextual models. These involve architectures such as skip gram- or transformer networks and can learn semantic representations [9, 11, 17–19].

Yet, simpler syntax-based representations still have their merits, too. They offer benefits when processing texts of morphologically rich languages such as German or Turkish [5], cope with infrequent or out-of-vocabulary words [3, 12], and can be trained on small corpora [7]. Last but not least, the underlying learning algorithms are easy to implement [6].

This note practically demonstrates the latter. In particular, we show that plain vanilla *Python* makes it super easy to implement code that computes intersection string kernels [14].

String kernel such as these have many practical applications. For instance, in combination with kernel PCA, they allow for embedding words into the space spanned by the feature space principal components of a given vocabulary. This can be used to visualize whole vocabularies (see Fig. 1) and seamlessly extends to out-of-vocabulary words.

Yet, in order to keep this note short, we leave the discussion of theory and practice of such applications to later.

The main part of this note is section 2 where we look at theory and practice alike. Indeed, for experienced *Python* programmers, it may be downright trivial to implement the techniques we discuss. Yet, as the underlying theory is not necessarily trivial, it seems appropriate to interlace our theoretical discussion with practical coding examples.

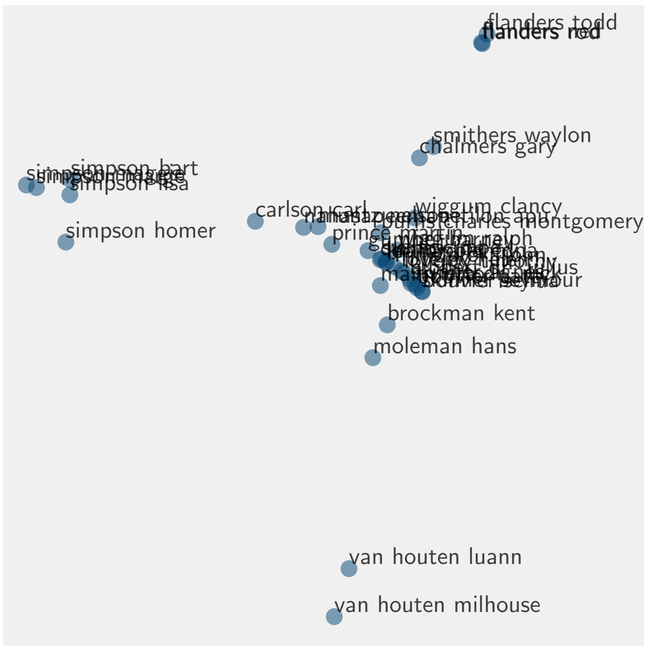
Another major part is found in the appendix where we show that intersection string kernels are actually Mercer kernels. However, the material presented there is not essential to readers who are mainly interested in the more practical aspects of the topic covered in this note.

As always, readers are expected to be passingly familiar with *Python*. Those who would like to experiment with the simple code snippets we provide only need to

```
from collections import Counter
```

bouvier patty	muntz nelson
bouvier selma	nahasapeemapetilon apu
brockman kent	prince martin
burns charles montgomery	qumby joe
carlson carl	riviera dr. nick
chalmers gary	simpson bart
flanders ned	simpson homer
flanders rod	simpson lisa
flanders todd	simpson maggie
frink prof. john	simpson marge
gumbel barney	skinner agnes
hibbert dr. julius	skinner seymour
krabappel edna	smithers waylon
leonard lenny	syslack moe
lovejoy helen	van houten luann
lovejoy timothy	van houten milhouse
mann otto	wiggum clancy
moleman hans	wiggum ralph

(a) vocabulary of 36 words



(b) words embedded in $\mathbb{R}^2$

Figure 1: A didactic vocabulary of words and 2D embeddings computed via KPCA on intersection string kernel matrices.

2 THEORY & PRACTICE

Given the venerable history of string kernels [15, 16], intersection string kernels [14] appeared surprisingly late in the game. The surprise is that they are essentially but histogram intersection kernels which have long been used in computer vision [1].

In this section, we discuss the underlying ideas and basic ingredients of intersection string kernels. Since *Python* makes it very easy to implement them, practical coding examples will be interlaced with our theoretical discussion. First, however, we need to recall several more basic concepts.

An alphabet $\mathcal{A}$ of size $|\mathcal{A}| = m$ is a set of symbols $\{a_1, \ldots, a_m\}$. Since our overriding interest lies in language processing, all our examples in this note will consider the alphabet

$$\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \ldots, \mathtt{z}, \text{␣}\} \tag{1}$$

of lower case Latin letters together with the character ␣ which denotes a blank space.

A string s over an alphabet $\mathcal{A}$ is a sequence of symbols $s \in \mathcal{A}^*$ where $\mathcal{A}^*$ is the **Kleene star** or set of all possible sequences (finite or infinite) of symbols in $\mathcal{A}$. If a string s is a sequence of l symbols, it is a string of length $|s| = l$. It is also an element of $\mathcal{A}^l \subset \mathcal{A}^*$ which is the set of all strings of length l over $\mathcal{A}$. If the size of $\mathcal{A}$ is m, then the size of $\mathcal{A}^l$ is m^l.

2.1 *n*-Grams

The multiset of ***n*-grams** (or n-spectrum) of a string s is the multiset $\mathcal{M}_n(s)$ of all its contiguous sub-strings of length n.

Since the meaning behind this definition becomes immediately apparent from looking at examples, let us consider these exemplary strings over the alphabet in (1)

$$s_1 = \texttt{"homer simpson"}$$
$$s_2 = \texttt{"lenny leonard"}$$
$$s_3 = \texttt{"ned flanders"}$$

Their multisets of bi-grams ($n = 2$) are

$$\mathcal{M}_2(s_1) = \{\texttt{"ho"}, \texttt{"om"}, \texttt{"me"}, \texttt{"er"}, \texttt{"r "}, \texttt{" s"}, \texttt{"si"}, \texttt{"im"}, \texttt{"mp"}, \texttt{"ps"}, \texttt{"so"}, \texttt{"on"}\}$$

$$\mathcal{M}_2(s_2) = \{\texttt{"le"}, \texttt{"en"}, \texttt{"nn"}, \texttt{"ny"}, \texttt{"y "}, \texttt{" l"}, \texttt{"le"}, \texttt{"eo"}, \texttt{"on"}, \texttt{"na"}, \texttt{"ar"}, \texttt{"rd"}\}$$

$$\mathcal{M}_2(s_3) = \{\texttt{"ne"}, \texttt{"ed"}, \texttt{"d "}, \texttt{" f"}, \texttt{"fl"}, \texttt{"la"}, \texttt{"an"}, \texttt{"nd"}, \texttt{"de"}, \texttt{"er"}, \texttt{"rs"}\}$$

and their multisets of tri-grams ($n = 3$) are

$$\mathcal{M}_3(s_1) = \{\texttt{"hom"}, \texttt{"ome"}, \texttt{"mer"}, \texttt{"er "}, \texttt{"r s"}, \texttt{" si"}, \texttt{"sim"}, \texttt{"imp"}, \texttt{"mps"}, \texttt{"pso"}, \texttt{"son"}\}$$

$$\mathcal{M}_3(s_2) = \{\texttt{"len"}, \texttt{"enn"}, \texttt{"nny"}, \texttt{"ny "}, \texttt{"y l"}, \texttt{" le"}, \texttt{"leo"}, \texttt{"eon"}, \texttt{"ona"}, \texttt{"nar"}, \texttt{"ard"}\}$$

$$\mathcal{M}_3(s_3) = \{\texttt{"ned"}, \texttt{"ed "}, \texttt{"d f"}, \texttt{" fl"}, \texttt{"fla"}, \texttt{"lan"}, \texttt{"and"}, \texttt{"nde"}, \texttt{"der"}, \texttt{"ers"}\}$$

Listing 1: computing the *n*-grams of a string *s*

```python
def n_grams(s, n):
    return map(''.join, zip(*[s[i:] for i in range(n)]))
```

Just to be clear, we reiterate that our examples in this note treat blank spaces as normal charters. That is, we understand strings such as "homer simpson" as a single word which happens to contain a blank.[1] Such blanks therefore occur in several of the above bi- and tri-grams.

We also point out that a **multiset** generalizes the notion of a set. Whereas the elements x of a set $\mathcal{X}$ must be distinct, i.e. only occur once, a multiset $\mathcal{Y}$ allows for multiple instances for each of its elements y. The number of occurrences of an element $y \in \mathcal{Y}$ is called its multiplicity and is denoted by $m_{\mathcal{Y}}(Y)$. For instance, for the bi-grams "le" and "en" in the above multiset $\mathcal{M}_2(s_2)$, we have

$$m_{\mathcal{M}_2(s_2)}(\texttt{"le"}) = 2$$
$$m_{\mathcal{M}_2(s_2)}(\texttt{"en"}) = 1$$

In *Python*, the natural data structure for multisets are lists. Even better, *Python* makes it ridiculously easy to compute n-grams of text strings. This is because it treats strings as iterable objects and provides a `str` class for convenient string processing. Our function `n_grams` in Listing 1 takes full advantage of these features and was previously explained in [2]. Using it, we can create n-gram lists for arbitrary strings. For instance

```python
for n in [2,3,4]:
    print (list(n_grams("homer simpson", n)))
```

results in

```
>>> ['ho', 'om', 'me', 'er', 'r ', ' s',
     'si', 'im', 'mp', 'ps', 'so', 'on']
>>> ['hom', 'ome', 'mer', 'er ', 'r s', ' si',
     'sim', 'imp', 'mps', 'pso', 'son']
>>> ['home', 'omer', 'mer ', 'er s', 'r si',
     ' sim', 'simp', 'imps', 'mpso', 'pson']
```

which agrees with what we would expect from our discussion up to this point.

Now that we know how to compute n-grams, let us have a closer look at, say, the bi-grams of our exemplary strings s_1, s_2, and s_3. In particular, let us see if there are any commonalities.

Apparently, s_1 and s_2 share the bi-gram "on". Moreover, s_1 and s_3 share the bi-gram "er" whereas s_2 and s_3 do not share any bi-gram at all.

Based on observations like these, it seems reasonable to say that strings s_1 and s_2 as well as s_1 and s_3 are somewhat similar. Strings s_2 and s_3, on the other hand, appear to be dissimilar.

In general, we could of course also resort to n-grams with $n > 2$ to come up with such statements about similarity. But how could or rather should we quantify the n-gram similarity of two any strings s_i and s_j ?

[1] Usually, we would treat strings with blank spaces as sequences of two or more words, say "homer" and "simpson". But, for this note, we deliberately decided not to do this.

Listing 2: computing the n-gram histogram of a string s

```
def n_gram_hist(s, n):
    return Counter(n_grams(s, n))
```

2.2 n-Gram Histograms

Our first step towards quantifying n-gram string similarities is to compute n-gram histograms of strings. As we shall see, the formal specification of this idea is more involved than its practical implementation. But let us be formal anyway.

To begin with, we recall that the purpose of a histogram is to count how often certain objects appear in some collection. In our case, the objects to be counted are n-grams and the collection they are to be counted in is the multiset of n-grams of a given string. There are (at least) two ways of formalizing the respective counting mechanism. We can think of the n-gram histogram $h_{n,s}$ of a string s over $\mathcal{A}$ as a discrete function

$$h_{n,s} : \mathcal{A}^n \to \mathbb{N} \tag{2}$$

or as a relation

$$h_{n,s} \subset \mathcal{A}^n \times \mathbb{N} \tag{3}$$

Both points of view have their merits for our purposes in this note, but, for now, we will only consider the second one.

This requires us to recall the notion of the support of a multiset which is but the underlying set of a multiset. In our context, the support set $\mathcal{S}_n(s)$ of the multiset $\mathcal{M}_n(s)$ of n-grams of a string $s \in \mathcal{A}^*$ is given by

$$\mathcal{S}_n(s) = \left\{ g \in \mathcal{A}^n \;\middle|\; m_{\mathcal{M}_n(s)}(g) > 0 \right\} \tag{4}$$

While this looks horrible, it simply means that $\mathcal{S}_n(s)$ is the set (not multiset!) of the n-grams $g \in \mathcal{A}^n$ that occur in s.

For any string $s \in \mathcal{A}^*$, we can determine $\mathcal{M}_n(s)$ and $\mathcal{S}_n(s)$. Given these, we can define the n-gram histogram $h_{n,s}$ of s as a relation or set of pairs

$$h_{n,s} = \left\{ \left(g, m_{\mathcal{M}_n(s)}(g) \right) \;\middle|\; g \in \mathcal{S}_n(s) \right\} \tag{5}$$

Again, this looks horrible but is nothing but a formal way of saying that our histogram pairs every n-gram in s with the number of times it occurs in s.

Working with *Python*, this view on n-gram histograms of strings is easily implemented, too. This is because the `collections` module provides the dictionary subclass `Counter` for counting hashable objects. Its use is shown in function `n_gram_hist` in Listing 2 and its effect is once again best understood by looking at examples.

For instance, the bi-gram histograms of two of our exemplary strings can be produced and printed using

```
for s in ["homer simpson", "lenny leonard"]:
    print (n_gram_hist(s, 2))
```

which results in

```
>>> Counter({'ho': 1, 'om': 1, 'me': 1, 'er': 1,
             'r ': 1, ' s': 1, 'si': 1, 'im': 1,
             'mp': 1, 'ps': 1, 'so': 1, 'on': 1})
>>> Counter({'le': 2, 'en': 1, 'nn': 1, 'ny': 1,
             'y ': 1, ' l': 1, 'eo': 1, 'on': 1,
             'na': 1, 'ar': 1, 'rd': 1})
```

Looking at these exemplary outputs, we observe that *Python* counters store objects to be counted as dictionary keys and their counts as dictionary values. Our exemplary outputs further indicate that `n_gram_hist` works as intended.

2.3 Intersection String Kernels

Now that we can practically compute n-gram histograms, we can take our second and final step towards quantifying the n-gram similarity of two strings s_i and s_j. Again, practice will be easier than theory.

To begin with, we introduce the shorthand

$$h_{n,s}[g] = m_{\mathcal{M}_n(s)}(g) \tag{6}$$

to denote the count of n-gram g in string s. Next, we let the set

$$\mathcal{I}_n = \mathcal{S}_n(s_i) \cap \mathcal{S}_n(s_j) \tag{7}$$

denote the intersection of the support sets of n-grams of s_i and s_j. With these definitions at hand, we then compute the function

$$k_n(s_i, s_j) = \sum_{g \in \mathcal{I}_n} \min\left\{ h_{n,s_i}[g],\, h_{n,s_j}[g] \right\} \tag{8}$$

which counts how many n-grams the two strings have in common.

To better understand the rationale behind the function or string similarity measure in (8), we consider another example involving simple bi-gram histograms and the following two strings

$$s_i = \texttt{"lenny leonard"}$$
$$s_j = \texttt{"hans moleman"}$$

Both these strings contain the bi-gram $g = \texttt{"le"}$. In fact, this is the only bi-gram they share so that $\mathcal{I}_2 = \{\texttt{"le"}\}$. In s_i, "le" occurs twice but in s_j it only occurs once. Hence, $h_{n,s_i}[g] = 2$, $h_{n,s_j}[g] = 1$, and $\min\{2, 1\} = 1$. In this example, the bi-gram similarity of strings s_i and s_j thus amounts to 1.

(For those who are interested in even more jargon: What we are computing in (8) is nothing but the size of the multiset intersection of $\mathcal{M}_n(s_i)$ and $\mathcal{M}_n(s_j)$. But let us leave it at that …)

Working with *Python*, the computation of (8) is a again a breeze. This is because the operator `&` allows for intersecting counters. That is, we neither have to worry about computing $\mathcal{I}_n$ nor about computing minima. For instance, to replicate parts of the example we just went through, we may use

```
hi = n_gram_hist("lenny leonard", 2)
hj = n_gram_hist("hans moleman", 2)
print (hi & hj)
```

This results in

```
>>> Counter({'le': 1})
```

which tells us that bi-gram "le" is shared once by our strings.

To replicate our example in full, i.e. to also sum over all the bi-grams $g \in \mathcal{I}_2 = \mathcal{S}_2(s_i) \cap \mathcal{S}_2(s_j)$, we simply use

```
print (sum((hi & hj).values()))
```

which yields

```
>>> 1
```

Listing 3: computing the intersection kernel of stings s_i, s_j

```
def intersection_str_kernel(si, sj, n):
    hi = n_gram_hist(si, n)
    hj = n_gram_hist(sj, n)

    return sum((hi & hj).values())
```

Summarizing all these computations in a single function leads to `intersection_str_kernel` as shown in Listing 3.

That is it! We now have a notion for the *n*-gram similarity of two strings and may use it in language processing applications.

However, for curious readers, there may remain a lingering open question, namely:

Q: Why have we been talking about intersection string *kernels*?

Or, to paraphrase a bit more technically:

Q: Why is our string similarity function in (8) called $k_n(\cdot, \cdot)$?

Well, to be brief, this is because of the very crucial fact that

A: Our string similarity function $k_n(\cdot, \cdot)$ is a Mercer kernel !

Proving this momentous claim is not that difficult but somewhat tedious. We therefore defer this to the appendix but encourage our readers to go through the arguments presented there. For now, we will look at a very basic practical application of what we just worked out.

2.4 Intersection String Kernel Matrices

In order to provide a first glimpse at what to practically do with intersection string kernels, we next compute $n \in \{2, 5\}$-gram similarity matrices for the 36 words in the vocabulary in Fig. 1(a).

To this end, we first of all assume that they are given in form of a list of strings

```
VOC = ["bouvier patty", ..., "wiggum ralph"]
```

Using this list, we then compute a list of *n*-gram histograms. Opting for $n = 2$, this can be accomplished by means of

```
n = 2
vocHists = [n_gram_hist(word, n) for word in VOC]
```

Using this list, we then compute a similarity matrix $S \in \mathbb{R}^{36 \times 36}$.

Note: The following way of implementing matrix S as a list of lists S is of course very, very bad practice! However, in this note, we deliberately opted not to use any *NumPy* functionalities (just to show that basic language processing can be done without it) …

```
m = len(VOC)
S = [[0] * m for i in range(m)]
for i, hi in enumerate(vocHists):
    for j, hj in enumerate(vocHists[i:], i):
        S[i][j] = sum((hi & hj).values())
        S[j][i] = S[i][j]
```

If we then print the resulting lists of lists, we obtain something as shown in Fig. 2, and, if we repeat the whole exercise with `n = 5`, we obtain a result as in Fig. 3.

Note: In order to improve readability, neither figure shows the numerous 0s contained in either similarity matrix.

What is strikingly apparent is that the 5-gram similarity matrix is much sparser than the 5-gram similarity matrix. This was to be expected, because, the longer an *n*-gram in a given word, the less likely it reoccurs in another word. We also observe the seven blocks or clusters along the main diagonals of both matrices. These reflect the syntactic similarities of the names of the members of the Bouvier, Flanders, Lovejoy, Simpson, Skinner, van Houten and Wiggum families.

All in all, these results are rather silly. Nevertheless, they indicate that *n*-gram similarities can reveal latent structures within a given vocabulary of words. More serious and more useful applications of *n*-gram similarities will be discussed in future notes.

```
12 7   1   1 1 1 1 1   1        1          3   1   1   2 3 1   1 1
 712 1 1   2 1 1 1   1 1 1   1  1 1 1 1 1  3   1   2 1 2 5 1   1 2
   112 1 1   1 2 1 1       1 1  3 3 1 2 1  1 1 1 1 2 2       1 3 3 1
 1 1 123 4 7 2 2 2   2 1   3 1 1  4 2 2 2  1 2 4 1 2 3 2 3 3 2   1 1
     1 411 1         1     2    1 1 3 2 1    4 3 3 3 4     1   1 1 1
 1 2   7 112 3 3 3   1 1   1      1   1 1  1 1 2     1 1 1 3         1
 1 1 1 2   311 8 8   1 1 1      1 1 2      2   1       2 2 3 1 1 1 2
 1 1 2 2   3 811 9 1   1        1 1        1   1       1 1 3 1 1 1 2 1
 1 1 1 2   3 8 912     1       12 1        1   1       1 1 3 1 1 1 2
 1   1         1  15   2     1 1        32 2           1 1   1
   1   2 1 1 1      12 1 3 1 1      2   12   3       1 1 2         2 2
 1 1   1   1 1 1 1 2 117                11 5 1 1 1     1 1 1     1
   1         1       3  13 1 1      1 3    1 1                       1
     1 3 2 1         1   112 2  1 1 1 2 1    2 1 2 1 2 1 1 1   3 1
   1 1 1           1 1   1 212 7  2 1 1  2     1           2   2 2
       1         1 1         7141 1   2 12   1 1 1 1 1   1 2 1
 1 1 3   1   1 1 2         1   18 3   2 1    1 1 1 2 2 1 1     3 2 1
   1 3 4 1 1 1 1 1         1 2 1311   4 1    1 2 1 2 2   1   1 4 3 1
   1 1 2 3   2       2   1 1 1     11 1    1 2 2 2 2 2 1 1 1
   1 2 2 2 1             3 2 1 22 4 121 2    2 2 3 3 3 1   2   1 2
   1 1 2 1 1       3 1 1   1   11 1   212  1 2     2 3 1 1   1   1 1
                   2 2 1     2 2         8                   1     1 1
 3 3 1 1   1 2 1 1 2   5 1          1   1 15   1   1   1 1 1 1       1
     1 2 4 1         3 1 1 2   11 1 2 2 2   11 7 7 7 8     1   1 1
 1 1 1 4 3 2 1 1 1     1   1 1 11 2 2 2    1 712 7 7 7 1 1 2   3 3
     1 1 3             1   2   11 1 2 3      7 711 7 7     1   2 1
 1 2 2 2 3                 1   12 2 2 3 2  1 7 7 713 9 1   1 1 1 2 1 1
   1 2 3 4 1         1     2   12 2 2 3 3    8 7 7 912     1 1 1 2
 2 2   2   1 2 1 1 1 1 1   1    1   1 1 1  1   1   1  12 7 1   1
 3 5   3   1 2 1 1 1 2 1   1   11 1 1   1  1   1       714 1 1 2 2
 1 1   3 1 3 3 3 3     1   1 2 2    1 2    1 1 2 1 1 1 1 114     1
     1 2     1 1 1 1           1  1     11 1       1 1   1  10   1 1
 1 1 3   1   1 1 1         3 2  3 4   1      1 3 2 1 1 1 2    1510 1
 1 2 3 1 1   1 1 1     1   1 2  2 3   2 1    1 3 1 2 2   2 1 11018 1
     1 1 1   2 2 2   2          1 1     11         1         1 1 112 6
           1   1     2   1               1 1       1               611
```

Figure 2: 2-gram similarity matrix for the words in Fig. 1(a).

```
 9 4
 4 9                                                       1
     9
      20
         8
           9
             8 5 5
             5 8 5
             5 5 9
                  12
                     9
                      14                     1
                        10
                           9
                             9 4
                             411
                                 5
                                   8
                                     8
                                      18
                                         9
                                           5
                       1                    12
                                               8 4 4 4 4
                                               4 9 4 4 4
                                               4 4 8 4 4
                                               4 4 410 6
                                               4 4 4 6 9
                                                         9 4
   1                                                     411
                                                            11
                                                               7
                                                                12 7
                                                                 715
                                                                     9 3
                                                                     3 8
```

Figure 3: 5-gram similarity matrix for the words in Fig. 1(a).

3 SUMMARY AND OUTLOOK

Machine learning for natural language processing is a corner stone for tasks such as intelligent document analysis [4, 8, 10, 13, 20, 21]. Many methods in this arena require numerical vector representations of words which are currently most commonly computed using neural networks [9, 11, 17–19]. However, light weight kernel methods have their merits, too [3, 6, 7, 12].

In this note, we had a first look at string kernels and studied the notion of intersection string kernels. We saw that they have one truly appealing property, namely ease of implementation. Our plain vanilla *Python* implementations for computing syntactic, i.e. n-gram based, string similarities merely involved 8 lines of code (2 lines in Listing 1, 2 lines in Listing 2, and 4 lines in Listing 3).

Moreover, the insight that an intersection string kernel is indeed a **Mercer kernel** will be of practical value, too, for it allows us to unleash everything we know about kernel machines on language processing. We will further elaborate on this in upcoming notes where we will study ideas such kernel PCA for word embeddings or kernel SVMs for language disambiguation.

APPENDIX

In this appendix, we work out why and how the string similarity function $k_n(s_i, s_j)$ defined in (8) is indeed a Mercer kernel. But we best start with a **disclaimer:** The constructions we present next are of purely theoretical interest. They are not meant to ever be implemented in practice. Rather, they are intended to prove that (8) has a hidden but important aspect to it.

To establish that $k_n(s_i, s_j)$ is a Mercer kernel, we will consider n-gram histograms from the point of view we did not follow up on in the main text of this note. That is, we will think of the n-gram histogram $h_{n,s}$ of a string s over $\mathcal{A}$ as a discrete function

$$h_{n,s} : \mathcal{A}^n \to \mathbb{N} \tag{9}$$

such that $h_{n,s}[g] = m_{\mathcal{M}_n(s)}(g)$ and, by convention, $h_{n,s}[g] = 0$ if g is not a substring of s.

We also recall that, for a finite alphabet $\mathcal{A}$ of size m, the finite size of $\mathcal{A}^n$ is m^n. This finiteness allows for a unique mapping between the $g \in \mathcal{A}^n$ and the numbers $r \in \mathcal{R} = \{1, 2, \ldots, m^n\} \subset \mathbb{N}$. For instance, for the set of tri-grams $\mathcal{A}^3$ over the alphabet in (1), we may consider the canonical assignment

$$\begin{aligned} \texttt{aaa} &\leftrightarrow 1 \\ \texttt{aab} &\leftrightarrow 2 \\ \texttt{aac} &\leftrightarrow 3 \\ &\vdots \end{aligned}$$

We may then write $g[i]$ to refer to the i-th n-gram in $\mathcal{A}^n$. This allows us to reconsider the n-gram histogram $h_{n,s}$ as a function

$$h_{n,s} : \mathcal{R} \to \mathbb{N} \tag{10}$$

where $h_{n,s}[r] = m_{\mathcal{M}_n(s)}(g[r])$ and, by convention, $h_{n,s}[r] = 0$ if $g[r]$ is not a substring of s.

Even more, since the domain $\mathcal{R}$ of this function is finite, we may identify $h_{n,s}$ with a vector $\boldsymbol{h}(s) \in \mathbb{R}^{m^n}$ where

$$\boldsymbol{h}(s) = \begin{bmatrix} h_{n,s}[1] \\ h_{n,s}[2] \\ \vdots \\ h_{n,s}[m^n] \end{bmatrix} \tag{11}$$

Note that vector $\boldsymbol{h}(s)$ does not need to carry a subscript $_n$ because this piece of information is implicitly encode in its dimensionality. Further note that we henceforth write

$$[\boldsymbol{v}]_r$$

to refer to the r-th entry of a vector $\boldsymbol{v}$.

Intersection String Kernels Are Mercer Kernels

Given what we just worked out, we observe that function $k_n(s_i, s_j)$ in (8) can just as well written as

$$k_n(s_i, s_j) = \sum_{r=1}^{m^n} \min\Big\{ \big[\boldsymbol{h}_i\big]_r, \big[\boldsymbol{h}_j\big]_r \Big\} \tag{12}$$

where

$$\boldsymbol{h}_i = \boldsymbol{h}(s_i) \tag{13}$$
$$\boldsymbol{h}_j = \boldsymbol{h}(s_j) \tag{14}$$

Now, if $k_n(s_i, s_j)$ was a Mercer kernel, it must also be an inner product in some latent, i.e. typically unknown, Hilbert space $\mathbb{H}$. In other words, if $k_n(s_i, s_j)$ was a Mercer kernel, we must be able to equivalently compute it as

$$k_n(s_i, s_j) = \boldsymbol{\varphi}_i^\intercal \boldsymbol{\varphi}_j \tag{15}$$

where

$$\boldsymbol{\varphi}_i = \boldsymbol{\varphi}(s_i) \tag{16}$$
$$\boldsymbol{\varphi}_j = \boldsymbol{\varphi}(s_j) \tag{17}$$

and $\boldsymbol{\varphi} : \mathcal{A}^* \to \mathbb{H}$ is an appropriate feature map that takes strings in $\mathcal{A}^*$ to vectors in $\mathbb{H}$.

Our problem at this point is thus find such a feature map. That is, to prove that $k_n(s_i, s_j)$ is a Mercer kernel, we need prove that there exists at least one function $\boldsymbol{\varphi} : \mathcal{A}^* \to \mathbb{H}$ such that (12) can equivalently be written as (15).

Looking at the systems of equations in (12)–(14) and (15)–(17), their structures appear encouragingly similar. That is, it does not seem impossible to establish a connection between them.

The main difficulty is that the right hand side of (12) involves non-linear functions (min) and therefore does not constitute an inner product. (If it did, our job was already done because we could simply let $\boldsymbol{\varphi}(s) = \boldsymbol{h}(s)$.)

However, the entries $[\boldsymbol{h}(s)]_r$ of the histogram vector $\boldsymbol{h}(s)$ are special in that they are counting numbers. And counting numbers can be represented using the following (not really practical but theoretically useful) bit-string format

$$\begin{aligned} 0 &= 0 \\ 1 &= 1 \\ 2 &= 11 \\ 3 &= 111 \\ 4 &= 1111 \\ &\vdots \end{aligned}$$

That is, we can represent each $[\boldsymbol{h}(s)]_r$ of $\boldsymbol{h}(s)$ either as a zero or as a sequence of ones.

To let all such sequences be of the same length, we may right-pad them with an appropriate amount of 0s. To this end, we note that a string of length l contains $l-n+1$ contiguous substrings or n-grams. Hence, we can identify each $[\boldsymbol{h}(s)]_r$ with a binary vector

$$z_r(s) = \Big[\overbrace{\underbrace{1\,1\,1 \cdots 1\,1}_{[\boldsymbol{h}(s)]_r}\ 0\,0\,0 \cdots\ 0\,0}^{l} \Big] \tag{18}$$

because

$$\big[\boldsymbol{h}(s)\big]_r = \sum_{q=1}^{l} \big[z_r(s)\big]_q \tag{19}$$

However, if we do this for two strings s_i and s_j, their lengths l_i and l_j may differ. A remedy is to consider a maximum length L longer than the length of any string we will ever encounter. For the point we are making here, the exact choice of L does not matter, it is really but a theoretical construct. When in doubt, we could chose it to be a ridiculously large number such as, say, $L = 10^{80}$, the number of atoms in the known universe. Assuming an appropriate L, we can represent each $[\boldsymbol{h}(s)]_r$ as

$$z_r(s) = \Big[\overbrace{\underbrace{1\,1\,1 \cdots 1\,1}_{[\boldsymbol{h}(s)]_r}\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \cdots\ 0\,0}^{L} \Big] \tag{20}$$

Next, we observe the following very peculiar property of binary numbers $x, y \in \{0, 1\}$, namely

$$\min\big\{x, y\big\} = x \cdot y \tag{21}$$

Just because of this special property of binaries, we can now actually write

$$\begin{aligned} \min\Big\{\big[\boldsymbol{h}(s_i)\big]_r, \big[\boldsymbol{h}(s_j)\big]_r\Big\} &= \min\left\{\sum_{q=1}^{L}\big[z_r(s_i)\big]_q, \sum_{q=1}^{L}\big[z_r(s_j)\big]_q\right\} \\ &= \sum_{q=1}^{L} \min\left\{\big[z_r(s_i)\big]_q, \big[z_r(s_j)\big]_q\right\} \\ &= \sum_{q=1}^{L} \big[z_r(s_i)\big]_q \cdot \big[z_r(s_j)\big]_q \\ &= z_r(s_i)^\intercal z_r(s_j) \end{aligned} \tag{22}$$

But this is finally to say that, if we consider a truly humongous vector $\boldsymbol{\varphi}(s) \in \mathbb{R}^{L \cdot m^n}$ with

$$\boldsymbol{\varphi}(s) = \begin{bmatrix} z_1(s) \\ z_2(s) \\ \vdots \\ z_{m^n}(s) \end{bmatrix} \tag{23}$$

we have found a feature vector representation $\boldsymbol{\varphi}(s)$ of a string s which allows us to write our n-gram similarity function as

$$k_n\big(s_i, s_j\big) = \sum_{r=1}^{m^n} \min\Big\{\big[\boldsymbol{h}_i\big]_r, \big[\boldsymbol{h}_j\big]_r\Big\} = \boldsymbol{\varphi}_i^\intercal \boldsymbol{\varphi}_j \tag{24}$$

This concludes our discussion because it establishes that $k_n(s_i, s_j)$ is indeed a Mercer kernel.

ACKNOWLEDGMENTS

This material was produced within the Competence Center for Machine Learning Rhine-Ruhr (ML2R) which is funded by the Federal Ministry of Education and Research of Germany (grant no. 01IS18038C). The authors gratefully acknowledge this support.

REFERENCES

[1] A. Barla, F. Odone, and A. Verri. 2003. Histogram Intersection Kernel for Image Classification. In *Proc. ICIP*.

[2] C. Bauckhage. 2015. NumPy / SciPy Recipes for Data Science: k-Medoids Clustering. researchgate.net. https://dx.doi.org/10.13140/2.1.4453.2009.

[3] M. Beeksma, M. van Gompel, F. Kunneman, L. Onrust, B. Regnerus, D. Vinke, E. Brito, C. Bauckhage, and R. Sifa. 2018. Detecting and Correcting Spelling Errors in High-quality Dutch Wikipedia Text. *Computational Linguistics in the Netherlands J.* 8 (2018), 122–137.

[4] D. Biesner, R. Ramamurthy, R. Stenzel, M. Lübbering, L. Hillebrand, A. Ladi, M. Pielka, R. Loitz, C. Bauckhage, and R. Sifa. 2022. Anonymization of German Financial Documents Using Neural Network-based Language Models with Contextual Word Representations. *Int. J. of Data Science and Analytics* 13, 2 (2022).

[5] P. Bojanowski, E. Grave, A. Joulin, and T. Mikolov. 2017. Enriching Word Vectors with Subword Information. *Trans. of the Association for Computational Linguistics* 5 (2017).

[6] E. Brito, B. Georgiev, D. Domingo-Fernandez, C.T. Hoyt, and C. Bauckhage. 2019. RatVec: A General Approach for Low-dimensional Distributed Vector Representations via Rational Kernels. In *Proc. KDML-LWDA*.

[7] E. Brito, R. Sifa, and C. Bauckhage. 2017. KPCA Embeddings: An Unsupervised Approach to Learn Vector Representations of Finite Domain Sequences. In *Proc. KDML-LWDA*.

[8] E. Brito, R. Sifa, C. Bauckhage, R. Loitz, U. Lohmeier, and C. Pünt. 2019. A Hybrid AI Tool to Extract Key Performance Indicators from Financial Reports for Benchmarking. In *Proc. Symposium on Document Engineering*. ACM.

[9] T. Brown and et al. 2020. Language Models Are Few-Shot Learners. In *Proc. NeurIPS*.

[10] C.L. Chapman, L.P. Hillebrand, M.R. Stenzel, T. Deusser, D. Biesner, C. Bauckhage, and R. Sifa. 2022. Towards Generating Financial Reports from Tabular Data Using Transformers. In *Proc. Cross Domain Conf. for Machine Learning and Knowledge Extraction (CD-MAKE)*.

[11] J. Devlin, M.-W. Chang, K. Lee, and K. Toutanova. 2019. BERT: Pre-training of Deep Bidirectional Transformers for Language Understanding. In *Proc. NAACL*.

[12] V. Gupta, S. Giesselbach, S. Rüping, and C. Bauckhage. 2019. Improving Word Embeddings Using Kernel PCA. In *Proc. Workshop on Representation Learning for NLP @ ACL*.

[13] L.P. Hillebrand, D. Biesner, C. Bauckhage, and R. Sifa. 2020. Interpretable Topic Extraction and Word Embedding Learning Using Row-Stochastic DEDICOM. In *Proc. Cross Domain Conf. for Machine Learning and Knowledge Extraction (CD-MAKE)*.

[14] R.T. Ionescu, M. Popescu, and A. Chaill. 2014. Can Characters Reveal Your Native Language? A Language Independent Approach to Native Language Identification. In *Proc. EMNLP*.

[15] C. Leslie, W. Eskin, and W.S. Noble. 2002. The Spectrum Kernel: A String Kernel for SVM Protein Classification. In *Proc. Pacific Symposium on Biocomputing*.

[16] H. Lodhi, C. Saunders, J. Shaw-Taylor, N. Christianini, and C. Watkins. 2002. Text Classification Using String Kernels. *J. of Machine Learning Research* 2 (2002).

[17] T. Mikolov, I. Sutskever, K. Chen, G. Corrado, and J. Dean. 2013. Distributed Representations of Words and Phrases and Their Compositionality. In *Proc. NIPS*.

[18] J. Pennington, R. Socher, and C. Manning. 2014. GloVe: Global Vectors for Word Representation. In *Proc. EMNLP*.

[19] M.E. Peters, M. Neumann, M. Iyyer, M. Gardner, C. Clark, K. Lee, and L. Zettlemoyer. 2018. Deep Contextualized Word Representations. In *Proc. NAACL*.

[20] M. Pielka, R. Sifa, L. Hillebrand, D. Biesner, R. Ramammurthy, A. Ladi, and C. Bauckhage. 2021. Tackling Contradiction Detection in German Using Machine Translation and End-to-End Recurrent Neural Networks. In *Proc. ICPR*.

[21] R. Ramamurthy, M. Pielka, R. Stenzel, C. Bauckhage, R. Sifa, T. Khameneh, U. Warning, B. Kliem, and R. Loitz. 2021. ALiBERT: Improved Automated List Inspection (ALI) with BERT. In *Proc. Symposium on Document Engineering*. ACM.