Applications of the Graph Tukey Depth

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Abstract

We study a recently introduced adaptation of Tukey depth to graphs and discuss its algorithmic properties and potential applications to mining and learning with graphs. In particular, since it is NP-hard to compute the Tukey depth of a node, as a first contribution we provide a simple heuristic based on maximal closed set separation in graphs and show empirically on different graph datasets that its approximation error is small. Our second contribution is concerned with geodesic core-periphery decompositions of graphs. We show empirically that the geodesic core of a graph consists of those nodes that have a high Tukey depth. This information allows for a parameterized deterministic definition of the geodesic core of a graph.

Keywords

graph Tukey depth, geodesic closure, closed set separations, geodesic core-periphery decomposition

1. Introduction

Centrality measures are of high importance in data analysis, as they typically capture the elements' "importance" quantitatively. Of course, the meaning of *importance* depends on the choice of the particular centrality measure. Different types of centrality measures have been introduced for networks (see, e.g., [1]), including *degree centrality*, *eigenvector centrality*, *Katz centrality*, *closeness centrality*, *betweenness centrality*, *page rank*, and *hubs and authorities*. In Fig. 1 we present a graphical illustration of some of these centrality measures for some small graphs for a visual comparison.

A relatively new centrality measure, the *graph Tukey depth*, has been introduced in [3]. It is based on the the original notion of Tukey depth that is defined over finite subsets of $\mathbb{R}^d$ [4, 5]. Informally speaking, the semantics of Tukey depth is as follows: An element e has *high* Tukey depth if it is "hard" to separate e from the rest of the finite ground set using separating hyper-planes only. Conversely, an element has *low* Tukey depth if it is "easy" to separate e from the rest of the set. In this context "hard" resp. "easy" means that the half-space bounded by the separating hyper-plane that contains e contains many resp. a few other elements of the finite ground set. Thus the elements' "importance" defined by Tukey depth relies on the possibility to separate it from other elements. This, however, directly connects the Tukey depth to machine learning approaches that are based on hyper-plane separations [6, 7]. For $\mathbb{R}^d$ and in other more general metric spaces [8], Tukey depth has been studied in the context of *machine learning*, in

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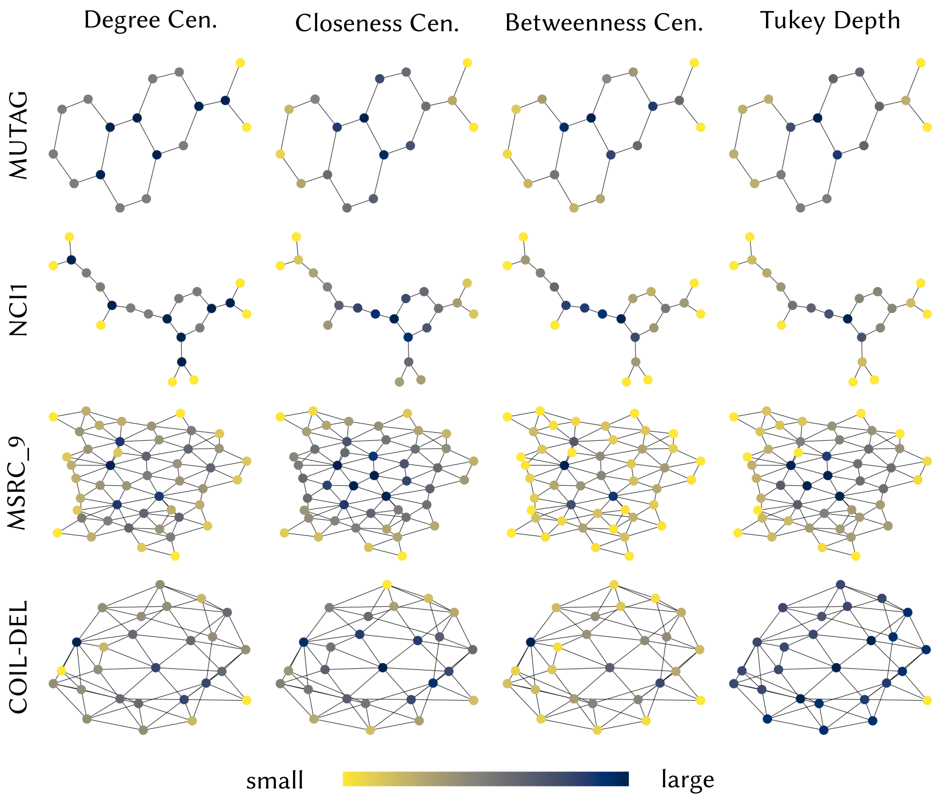


Figure 1: The *Degree Centrality*, *Closeness Centrality*, *Betweenness Centrality* and *Tukey Depth* of nodes in graphs selected from different graph datasets [2]. The centrality (resp. depth) values are *normalized* (i.e, mapped to the interval $[0, 1]$) by their maximum values in the graph. In particular, nodes of the smallest (resp. highest) centrality values are denoted by yellow (resp. blue).

particular, for object classification [9]. In the case of learning linear classifiers, it is shown in [10] that the *Tukey median*, i.e., the points with the highest depth, are related to the Bayes point.

An advantage of Tukey depth over other measures is that the *exact* geometric position of points in $\mathbb{R}^d$ is not relevant for their depth. Thus, the Tukey depth is relatively stable regarding outliers and is therefore used for outlier detection [11, 12]. Moreover, as Tukey depth relies on separations it is possible to adapt it to other domains that allow hyper-plane respectively half-space separations. This is how the original Tukey depth is adapted to geodesic closure systems over graphs [3]. The exact notions of graph Tukey depth and geodesic closures can be found in Sec. 2.

Similarly to the fact that the computation of the Tukey depth in $\mathbb{R}^d$ is NP-hard [13], it is also NP-hard to compute the graph Tukey depth of a node [3]. Motivated by this negative result, one of our main contributions is a *heuristic* algorithm for *approximating* graph Tukey depth. It runs in time polynomial in the size of the input graph and approximates the Tukey depth of a node with one-sided error by *overestimating* it. Our experimental results with small graphs clearly demonstrate that the approximation is close to the exact Tukey depth, by noting that for larger graphs we were not able to evaluate the approximation performance of our algorithm, as it was *not* possible to calculate the *exact* Tukey depth in practically feasible time.

Our heuristic is based on our greedy algorithm designed in [14] for solving the more general maximal closed set separation (*MCSS*) problem. In the particular case of graphs, the problem is as follows: Given two subsets A, B of the node set of a graph, find two (inclusion) maximal disjoint geodesically closed node sets that separate A and B from each other, i.e., that particularly contain A resp. B. As maximal disjoint closed sets provide a separation of elements it is therefore natural to ask the following question: *Is there a connection between graph Tukey depth and node separation with geodesically closed node sets?* We give an affirmative answer to this question by showing experimentally that solving the *MCSS* problem gives a good approximation quality of the exact Tukey depth in case of small graph datasets. This mainly relies on the following fact: For any set containing at least one node of high Tukey depth, there exists *no* large disjoint closed set.

It follows from the definition of graph Tukey depth that it is related to other concepts based on geodesic convexity. One of these notions is the recent *probabilistic* definition of *geodesic core-periphery decomposition* of graphs. It was introduced in [15] and studied in [15, 16, 17, 18]. Hence, our second question is concerned with the following problem: *Is there a connection between graph Tukey depth and geodesic core-periphery decompositions?* The geodesic core-periphery decomposition breaks up some types of graphs (including interaction graphs such as social networks) into a *dense* core and a *sparse* "surrounding" periphery [15] (see Fig. 3 for a visual example). While some graphs (e.g., Erdős-Rényi, Barabási-Albert, and Watts-Strogatz random graphs) seem to have no periphery, others (e.g., trees and fully connected graphs) seem to have no core. Since this behavior is not well-understood up to now we will use the Tukey depth to find out what is really happening. It seems that the geodesic core of a graph consists of those nodes in the graph that are of high Tukey depth (see Fig. 4 for some examples). This observation allows for a *parameterized deterministic* definition of the cores. That is, the core of a graph can be defined by those nodes that have a Tukey depth greater than a user specified threshold. Our empirical results clearly demonstrate that using the right threshold, the probabilistic definition of cores in [15] *coincides* with our deterministic one.

The rest of the paper is organized as follows. In Sec. 2 we first collect necessary notions and notations. In Sec. 3 we present our heuristic for approximating the Tukey depth and evaluate it empirically on small graph datasets. Sec. 4 contains some examples showing that graph Tukey depth is strongly related to existing mining and learning algorithms on graphs that rely on graph geodesic convexity. Finally, in Sec. 5 we mention some open questions for future research.

2. Preliminaries

In this section, we collect the necessary notions and fix the notation. For a graph $G = (V, E)$, $V(G)$ and $E(G)$ denote the set V of nodes and the set E of edges, and n and m stands for $n = |V(G)|$ and $m = |E|$, respectively. Unless otherwise stated, by graphs we always mean undirected graphs without loops and multi-edges. For any $u, v \in V(G)$, the (geodesic) interval $[u, v]$ is the set of *all* nodes on *all* shortest paths between u and v (see Fig. 2a for an example). A set of nodes $X \subseteq V(G)$ is called (geodesically) closed iff for all $u, v \in V(G)$, $u, v \in X$ implies $[u, v] \subseteq X$. The closure $\rho(X)$ of a set $X \subseteq V(G)$ is the smallest closed set containing X (see

(a) (b)

Figure 2: This figure shows (a) the geodesic interval $[u, v]$, i.e., the nodes on all shortest paths between u, v (blue) and (b) the geodesic closure $\rho(\{u, v\})$, i.e., the smallest set of nodes that contains u, v and all nodes on all shortest paths between arbitrary node pairs of the set (red).

Fig. 2b for an example of $\rho(\{u, v\})$).

Graph Tukey Depth For a graph G the *Tukey depth* of a node $v \in V(G)$ is defined as follows [3]: Let $C \subset V(G)$ be a closed set of *maximum* cardinality such that $v \notin C$. The *Tukey depth* of v, denoted by $\mathrm{td}(v)$ is defined by $\mathrm{td}(v) = |V(G)| - |C|$. The definition implies that the larger a closed set which does not contain v, the smaller its depth is.

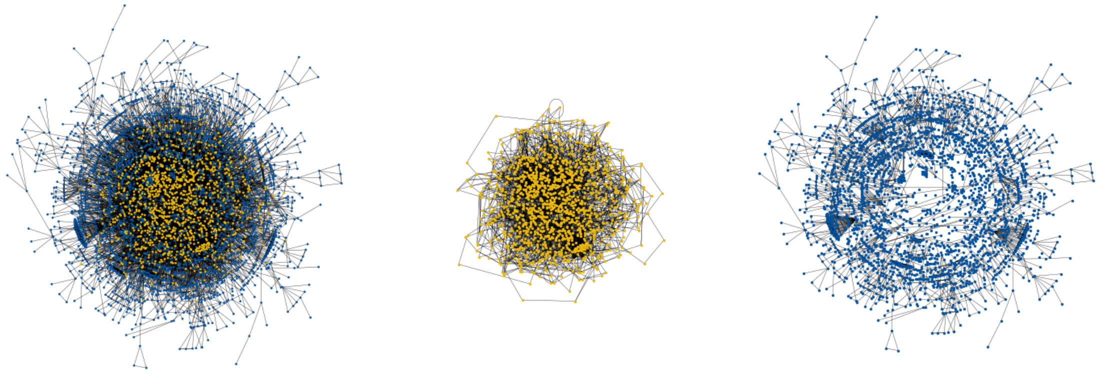
Geodesic Cores Up to now, *geodesic cores* [15] are defined *probabilistically* only. Informally, the geodesic core of a graph consists of those nodes which are contained in "every" geodesic closed set that is generated by a small number of random nodes. Of course, the core defined in this way can be empty, but it turns out that this is not the case for interaction graphs (e.g. social networks) [15]. Adapting the definition in [15] slightly, we define the geodesic core of a graph G, denoted by $\mathcal{C}$ as follows. Let $X_1, X_2, \ldots$ be a sequence of sets where each set consists of $k > 0$ nodes selected independently and uniformly at random from $V(G)$. Then $\mathcal{C} = \bigcap_{j=1}^{i} \rho(X_j)$, where i is the *smallest* integer satisfying $\bigcap_{j=1}^{i} \rho(X_j) = \bigcap_{j=1}^{i+1} \rho(X_j)$ is the core of G. Obviously, this definition is not deterministic since different choices of X_j and of k can lead to different cores. Nevertheless, the experiments in [16] with large real-world networks show that for $k \approx 10$, the core (if it exists) does *not* depend on the particular choice of the generator elements. The *core-periphery decomposition* of a graph is composed of the subgraph induced by the core nodes and that by the remaining nodes, called *periphery*. In Fig. 3 we give a visual example of the core-periphery decomposition of the CA-GrQc network [19][1].

MCSS problem for Graphs We present here the *MCSS* problem restricted to geodesic closures over graphs (see [14] for the generic definition). Given a graph $G = (V, E)$ and two node sets $A, B \subseteq V(G)$ then the *MCSS* problem is to find two geodesic closed sets $A', B' \subseteq V(G)$ with $A' \cap B' = \emptyset$ and $A \subseteq A', B \subseteq B'$ that are maximal, i.e., there exist no proper supersets of A' resp. B' that fulfill the above properties.

3. Approximating the Tukey Depth

Motivated by the negative complexity result concerning the calculation of Tukey depth, in Sec. 3.1 below we first propose a *heuristic* based on the maximal closed set separation (*MCSS*)

[1] This network is build by the co-authorships in the general relativity and quantum cosmology community.



(a) Entire Network (b) (Geodesic) Core (c) Periphery

Figure 3: Example of a geodesic core-periphery decomposition (c.f. [16]) (a) the CA-GrQc network [19] with core in orange and periphery in blue, (b) its (geodesic) core, (c) its periphery.

algorithm in [14] that solves the *MCSS* problem (Sec. 2). It approximates Tukey depths with one-sided error. We then show experimentally on different types of *small* graphs that the results obtained by our heuristic are *fairly close* to the exact ones. Furthermore, our algorithm is, even on small graphs, up to 200 times faster than the exact one (Sec. 3.2). It is important to emphasize that it was not possible to calculate the exact Tukey depths for larger graphs in a feasible time.

3.1. The Heuristic

Recall that the *exact* Tukey depth of a node v is defined by $\mathrm{td}(v) := |V(G)| - |C|$, where $|C|$ is the maximum cardinality of a closed set C not containing v. It can be computed exactly using an *integer linear program* (see [3] for the details). The computationally *hard* part of the problem is to find a closed set of *maximum* size. Our heuristic addresses this problem by considering an inclusion *maximal* closed set only, instead of a maximum sized closed set. This relaxation, which distorts of course the exact value of Tukey depth, allows us to apply the efficient greedy algorithm proposed in [14] for solving the maximal closed separation problem. In what follows, for any $v \in V(G)$, $\widehat{\mathrm{td}}(v)$ denotes the approximation of $\mathrm{td}(v)$ obtained with our heuristic.

Given a graph G, the rough idea to approximate the Tukey depth of a node $v \in V(G)$ is to find an inclusion maximal geodesically closed set $C \subseteq V(G)$ with $v \notin C$. Such a set C can be found by applying the *MCSS* algorithm [14] with nodes v, v' as input. Then the output of the algorithm is a solution to the *MCSS* problem, i.e., it consists of two node sets $H_v, H_{v'} \subseteq V(G)$ with $v \in H_v$ and $v' \in H_{v'}$ that are disjoint, closed, and inclusion maximal concerning these properties. That is, there exists no proper supersets of $H_v, H_{v'}$ with the same properties. The Tukey depth can then be approximated using the cardinalities of H_v resp. $H_{v'}$. For a fixed node v, the result depends on the particular choice of v'. To improve the approximation quality, we therefore call the *MCSS* algorithm for each node v several times with *different* nodes $v' \neq v$.

The pseudo-code of the above heuristic is given in Alg. 1. In Line 1 we initialize the Tukey depth of all nodes in G by setting them to the maximum possible value, i.e., to $|V(G)|$. We repeat

Algorithm 1: Approximation of Graph Tukey Depth

Input : graph G

Output : approximation $\widehat{\mathrm{td}}(v)$ of $\mathrm{td}(v)$ for all $v \in V(G)$

1 $\widehat{\mathrm{td}}(v) \longleftarrow |V(G)|$ for all $v \in V(G)$;
2 **for** $v \in V(G)$ **do**
3   **for** $v' \in \Gamma(v)$ **do**
4         $H_{v'}, H_v = MCSS(\{v'\}, \{v\})$;
5   **for** $x \in V(G)$ **do**
6   **if** $x \notin H_{v'}$ **then**
7                 $\widehat{\mathrm{td}}(x) = \min\{\widehat{\mathrm{td}}(x), |V(G)| - |H_{v'}|\}$;
8   **if** $x \notin H_v$ **then**
9                 $\widehat{\mathrm{td}}(x) = \min\{\widehat{\mathrm{td}}(x), |V(G)| - |H_v|\}$;
10 **return** $\widehat{\mathrm{td}}(v)$ for all $v \in V(G)$

the procedure described above for all nodes $v \in V(G)$ and all their neighbors $v' \in \Gamma(v)$ (see the for-loops in Line 2 and 3). In this way we solve the *MCSS* problem for all input sets $\{v\}, \{v'\}$, i.e., we separate v from all its neighbors v' by maximal disjoint closed sets $H_v, H_{v'}$ (see Line 4). Note that the Tukey depth of a node x is based on closed set of maximum cardinality not containing x. Thus, that assuming the node x does not lie in the set H_v then the cardinality of H_v is smaller or equal than a closed set of maximum cardinality not containing x. Hence, we can update the current Tukey depth approximation of *all* graph nodes $x \in V(G)$ as follows: Take the minimum over the old and the new approximation which is the cardinality of $V(G) \setminus H_v$ if $x \notin H_v$ or $V(G) \setminus H_{v'}$ if $x \notin H_{v'}$ (see Line 7 and Line 9).

By construction, Alg. 1 finds only *maximal* and *not* maximum closed sets, resulting in an one-sided error in the estimate of Tukey depths. This is formulated in the proposition below.

Proposition 1. *Alg. 1 overestimates the Tukey depth, i.e., for the output* $\widehat{\mathrm{td}}(v)$ *returned by Alg. 1 we have* $\widehat{\mathrm{td}}(v) \geq \mathrm{td}(v)$*, for all* $v \in V(G)$*.*

Regarding the runtime of Alg. 1, note that the inner for loop starting with Line 4 is executed $O(m)$ times because we iterate over all neighbors (i.e. all edges are considered twice). The runtime of the inner loop (Lines 3–9) is dominated by the *MCSS* algorithm which calls the closure operator at most $O(n)$ times [14]. Using that the geodesic closure can be determined in time $\mathcal{O}(m \cdot n)$ [20] we have the following result for the total runtime of Alg. 1:

Proposition 2. *Alg. 1 outputs an upper bound of the Tukey depth for all nodes of G in $\mathcal{O}(m^2 \cdot n^2)$ time.*

The runtime of the approximation algorithm can be improved by considering for each node v a *fixed* number of distinct nodes v', or by considering a fixed subset $W \subseteq V(G)$, instead of the whole node set $V(G)$ in the outer loop (see Lines 2–9). It is left to further research to analyze how these changes affect the quality of the approximation performance.

3.2. Experimental Evaluation

In this section we empirically evaluate the approximation quality and runtime of Alg. 1 on datasets containing *small* graphs[2]. Regarding the approximation quality, we compare the results obtained by our algorithm to the exact Tukey depths computed with the algorithm in [3]. For the evaluation we consider 19 graph datasets of different types (small molecules, small graphs from bioinformatics and computer vision, and small social networks) from [2] (see columns 2–4 of Tab. 1 for the number of graphs and their average number of nodes and edges). The *average size* of the graphs ranges from 14 (*PTC_MM*) up to 82 (*OHSU*); their *average edge numbers* from 14 to 200. The reason for considering small graphs only is that the exact algorithm from [3] was unable to calculate the Tukey depth for larger graphs in less than one day (see the last two columns). For practical reasons, we removed all disconnected graphs from the original datasets, by noting that our heuristic works for disconnected graphs as well.

The results are presented in Tab. 1. It contains the approximation qualities measured in different ways (columns 5–10) and the runtime of the exact (column 11) and our heuristic algorithm (column 12). The datasets are sorted according to their *absolute approximation error* (column 5 of Tab. 1), i.e., the sum of all differences between the approximation and the exact Tukey depth over all nodes and all graphs in the dataset.

Regarding the *absolute error*, our approximation results are equal to the exact Tukey depths for 5 out of the 19 datasets, while their computation was faster by a factor of up to 100 (see row *PTC_MM*). Our algorithm has the largest absolute error of 4155 on the *COIL-DEL* graphs, by noting that this dataset consists of 3900 graphs. Hence, the error per graph is only slightly above one. Additionally, we look at the *relative errors* (column 6), i.e., the absolute error divided by the sum of all depths. We use this measure to validate that our algorithm performs very well, by noting that the relative errors are below $4 \cdot 10^{-3}$ for all graph datasets. The *per node error* (column 7) is the average error our algorithm makes per node, while the *per graph error* (column 8) is the error it has on average per graph. Regarding the *per node error*, the worst case is for the *COIL-DEL* dataset (last row) with an average error of 0.05. For the *per graph error*, the worst result has been obtained for the *OHSU* dataset, where the approximation overestimates the sum of all node depths by 1.65 per graph on average. This shows that our approximation algorithm performs very well, especially, if considering the averages over the datasets.

Finally, we studied also the worst case approximations for nodes and graphs. In particular, the columns *Max. Node Error* resp. *Max. Graph Error* denote the *maximum error* of the algorithm on single nodes resp. single graphs. The results show that there is a very low error of at most 3 per node for 13 out of the 19 datasets. For three graph datasets, the *maximum error per node* is at most 7 and we have a maximum error between 11 and 19 in three cases. Regarding the *maximum error per graph*, a similar behavior can be observed by noting that except for *OHSU* and *Peking_1*, the *maximum node errors* and *maximum graph errors* are close to each other. This implies that there are only a *few* nodes with a *high* approximation error. It is an interesting question to pinpoint the properties of such nodes and graphs that are responsible for the high approximation errors. The last two columns show the *runtimes* of the two algorithms. Our algorithm (last column) is faster than the exact one by at least one order of magnitude.

[2] See https://github.com/fseiffarth/AppOfTukeyDepth for the code.

Data	Graph Number	Avg. Nodes	Avg. Edges	Error (absolute)	Error (relative)	Error per Node	Error per Graph	Max. Node Error	Max. Graph Error	Exact Run-time (s)	Approx. Run-time (s)
BZR	405	35.75	38.36	0	0	0	0	0	0	56.60	1.04
PTC_MM	336	13.97	14.32	0	0	0	0	0	0	21.09	0.21
COX2	467	41.22	43.45	0	0	0	0	0	0	76.10	1.27
Cuneiform	267	21.27	44.80	0	0	0	0	0	0	2.00	0.61
DHFR	756	42.43	44.54	0	0	0	0	0	0	266.33	3.19
PTC_FR	351	14.56	15.00	1	4.50e-05	1.96e-04	2.85e-03	1	1	23.81	0.25
PTC_FM	349	14.11	14.48	1	4.80e-05	2.03e-04	2.86e-03	1	1	20.64	0.22
MUTAG	188	17.93	19.79	1	6.50e-05	2.97e-04	5.32e-03	1	1	3.01	0.09
PTC_MR	344	14.29	14.69	2	9.20e-05	4.07e-04	5.81e-03	1	1	23.29	0.23
KKI	83	26.96	48.42	12	1.01e-03	5.36e-03	1.45e-01	2	4	40.45	2.43
IMDB-BINARY	1000	19.77	96.53	19	4.37e-04	9.61e-04	1.90e-02	1	3	723.07	113.14
NCI1	3530	29.27	31.88	34	5.20e-05	3.29e-04	9.63e-03	6	8	1194.63	7.76
Peking_1	85	39.31	77.35	40	1.30e-03	1.20e-02	4.71e-01	3	10	4761.33	48.08
MSRC_21C	209	40.28	96.60	86	1.28e-03	1.02e-02	4.11e-01	12	12	51.78	3.06
MSRC_9	221	40.58	97.94	89	1.21e-03	9.92e-03	4.03e-01	7	8	49.33	2.92
OHSU	79	82.01	199.66	130	8.54e-04	2.01e-02	1.65e+00	2	13	42887.32	235.40
ENZYMES	569	31.68	61.44	307	1.68e-03	1.70e-02	5.40e-01	7	10	933.79	8.25
MSRC_21	563	77.52	198.32	877	1.39e-03	2.01e-02	1.56e+00	19	25	3679.24	58.98
COIL-DEL	3900	21.54	54.24	4155	4.05e-03	4.95e-02	1.07e+00	11	17	2242.05	43.00

Table 1
Graph data of different sizes selected from [2]. Disconnected graphs are removed from the original datasets. The columns regarding the approximation quality denote the following. *Error (absolute)* denotes the overall error on the dataset, *Error (relative)* denotes the relative error regarding the depths, *Error per Node* denotes the average error per node, *Error per Graph* denotes the average error per graph, *Max. Node Error* denotes the maximum error for a node and *Max. Graph Error* denotes the maximum error on a graph. The last two columns show the runtimes of the exact and approximation algorithm in seconds.

In summary, the evaluation of Alg. 1 clearly shows that our heuristic performs well in approximating the graph Tukey depth. It is faster (up to 200 times) than the exact algorithm, even on small graph datasets. Regarding larger graphs, this gap in runtime will increase because of the exponential runtime of the exact algorithm. Additionally, the very small relative errors (at most $4 \cdot 10^{-3}$), the average errors (at most 1.65 per graph), and also the worst case errors show that the algorithm can be used for further applications based on the Tukey depth (see Sec. 4).

4. Applications to Mining and Learning in Graphs

This section deals with the connection of *Tukey depth* to *node separability* and to *geodesic core-periphery decompositions.* We first state three important properties of Tukey depth. In particular, Proposition 3 clarifies the role of Tukey depth in the context of geodesic closed sets. Propositions 4 and 5 are from [3].

Proposition 3. *Let G be a graph, $v \in V(G)$ with $\mathrm{td}(v) = n - c$, and $C \subseteq V(G)$ a geodesically closed node set with $|C| > c$. Then $v \in C$.*

Proposition 4. *Let G be a graph, $X \subseteq V(G)$, and C be the geodesic closure of X. Then the Tukey depth is a quasi-concave function, i.e., for all $c \in C$ we have $td(c) \geq \min\{td(x) : x \in X\}$.*

Proposition 5. *Let G be a graph, $k \in \mathbb{N}$, and $X = \{v \in V(G) : \mathrm{td}(v) \geq k\}$. Then X is geodesically closed.*

To underline the importance of these three statements, we give two examples that show how they influence existing concepts based on geodesic closures.

Example 1: Node Classification and Active Learning In [21, 14, 22, 23], disjoint half-spaces and closed sets are used for binary classification in closure systems, for node classification, and active learning in graphs using geodesic convexity. Given the Tukey depth $\mathrm{td}(v)$ of a node v, Proposition 3 immediately implies that a separating half-space or closed set not containing v cannot have a cardinality greater than $n - \mathrm{td}(v)$. Thus, for nodes of *high* Tukey depth there is *no* large geodesic closed set *not* containing them. Hence, Proposition 3 implies a nice theoretical connection between Tukey depth and the maximum size of separating half-spaces and closed sets. Using approximate Tukey depths, the predictive performance of all the above methods can possibly be improved.

Example 2: Geodesic core-periphery decomposition The geodesic core-periphery decomposition of graphs was analyzed in [16, 15, 17]. In particular, it was found in [15, 17] that many social networks consist of a dense geodesic core "surrounded" by a periphery (see Fig. 3 for an example). While some graphs, especially tree-like graphs, seem to have no core, others, such as graphs sampled from random models like Erdős-Rényi, Barabási-Albert and Watts-Strogatz seem to have no periphery. Moreover, the closure of a small number of randomly chosen graph nodes (≈ 10) always contains the geodesic core (if it exists). Furthermore, if the nodes are sampled from the geodesic core only, then the closure of the nodes is the geodesic core itself. If

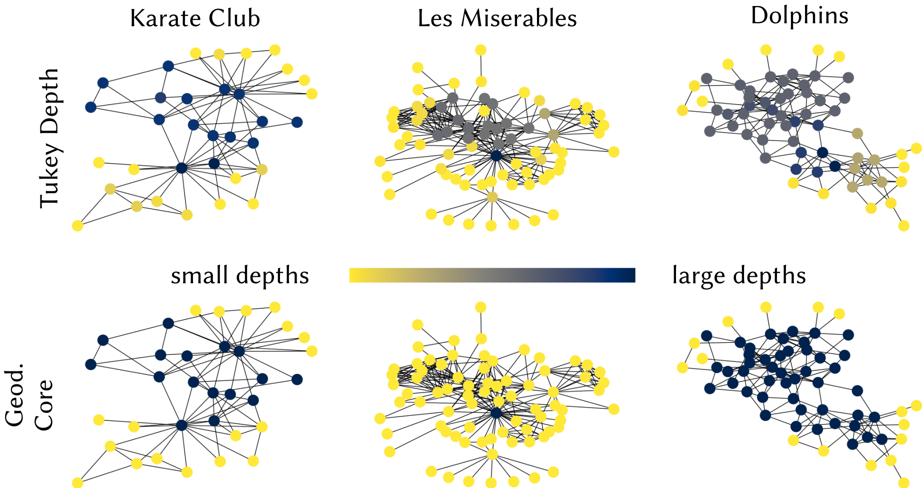


Figure 4: Tukey depth (top) vs. geodesic core-periphery decomposition (bottom) for the Karate Club [24], Les Miserables character [25], and Dolphins social networks [26]. For the different Tukey depths we use sequential colors. Core and periphery nodes are denoted by blue and yellow, respectively.

we compute the closure of, say, 10 randomly chosen nodes from the entire network (Fig. 3a), then the closure always contains the core (orange nodes in Fig. 3b). If all random nodes belong to the core (orange nodes in Fig. 3a), then their closure is the core itself. The above statements explain this behavior. Using that the core is always contained in the closure of a small number of randomly chosen nodes, from Proposition 4 it follows that the nodes in the core are those with the highest Tukey depths. Moreover, the quasi-concaveness implies that if the core is generated by a few nodes from the core, then the core nodes must have a very close Tukey depth. Finally, using Proposition 5, we have that the set of nodes in a graph with a Tukey depth above some threshold is always closed; cores arise as a special case of this property. These three properties motivate the following *deterministic* definition of geodesic cores:

Definition 1. *The k-geodesic core of a graph G is defined by*

$$C_k := \{v \in V(G) : \mathrm{td}(v) \geq k\} \ .$$

To empirically confirm our claim that the core contains the nodes with the highest Tukey depths, consider the three graphs in Fig. 4. For each graph, we computed the exact Tukey depths (top) and their geodesic cores (bottom). For the Karate Club network (left) considered also in [3], the core exactly matches the nodes of the highest Tukey depth. Furthermore, there is not much fluctuation in the depths of the core nodes. In fact, all nodes of Tukey depth of at most 3 belong to the periphery and all nodes of Tukey depth 19 or 21 to the core. In the case of the Les Miserables character network (middle), there is only a single node with a very high Tukey depth of 57, surrounded by nodes of depth less than 35. In this case, the core algorithm returns only the node with the highest Tukey depth, showing that the graph Tukey depth can possibly be used to improve core-periphery decomposition. This is also the case for the Dolphin community graph (right), where the core consists of nodes with Tukey depth greater than 2, while all nodes in the periphery have a Tukey depth of at most 2.

5. Concluding Remarks

Our results indicate that graph Tukey depth is an interesting and promising concept for mining and learning with graphs. The study of the relationship of graph Tukey depth to other node centrality measures is an interesting question for further research (see Fig. 1). For example, while the centroid(s) in trees [27, 28] are exactly the nodes with the highest Tukey depth, this is not necessarily the case for more general graphs beyond trees.

Another important issue is to understand the semantics of graph Tukey depth better. For example what are the properties of the nodes with the highest depth (cf. the def. of Tukey-median in $\mathbb{R}^d$)? We have empirically demonstrated that graph Tukey depth can closely be approximated for small graphs. It is a question of whether this result holds for (very) large graphs as well. To answer this question, the scalability of our approximation algorithm should be improved on the one hand. On the other hand, one needs (possibly tight) theoretical upper bounds on graph Tukey depths. Another interesting question is to identify graph classes for which our approximation is always exact. While this is the case for trees, it is unclear whether it holds for outerplanar graphs as well. We believe that this question can be answered affirmatively by using techniques from [16]. As shown in the paper, graph Tukey depth "naturally" connects different concepts based on geodesically closed sets; examples include the deterministic definition of k-geodesic cores. This implies that using our fast core approximation algorithm [16], we can closely approximate the set of nodes with the highest Tukey depth.

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