Maximum Margin Separations in Finite Closure Systems

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Hyperplane Separation in $\mathbb{R}^d$

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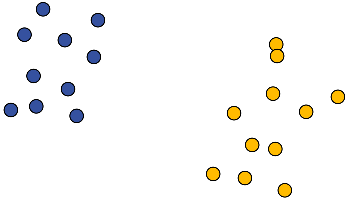

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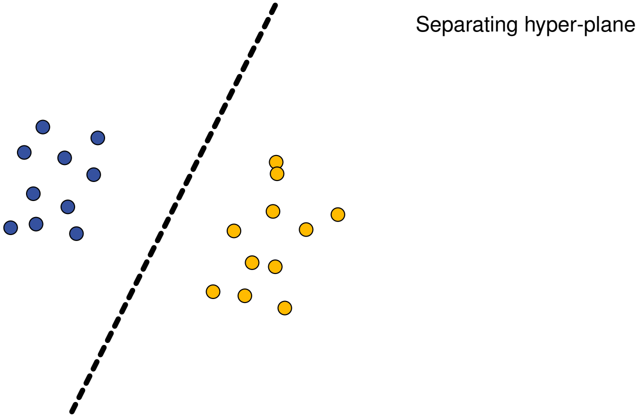

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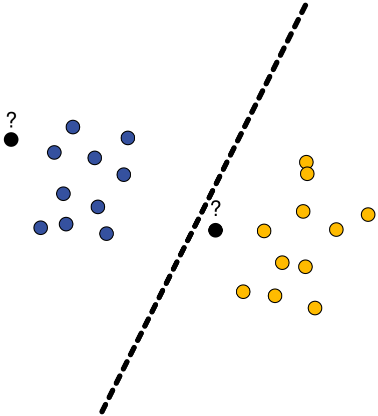

Separating hyper-plane **(arbitrary)**

- classical machine learning problem (Rosenblatt, 1958), not unique, over-fitting

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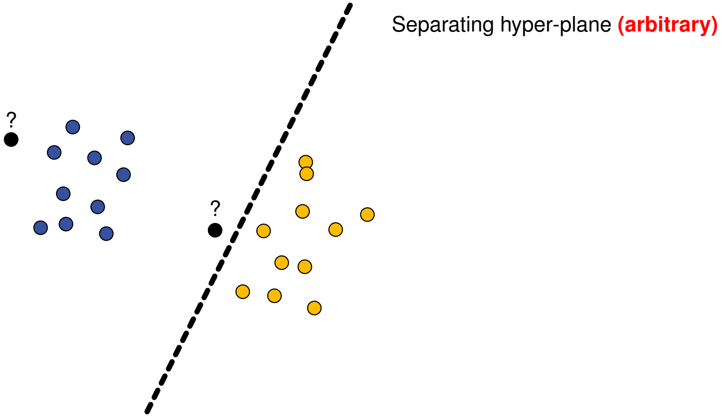

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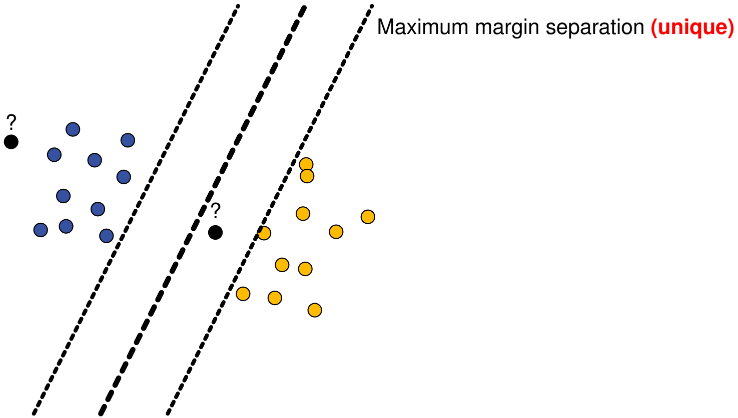

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- support vector machines (Vapnik, 1992), unique

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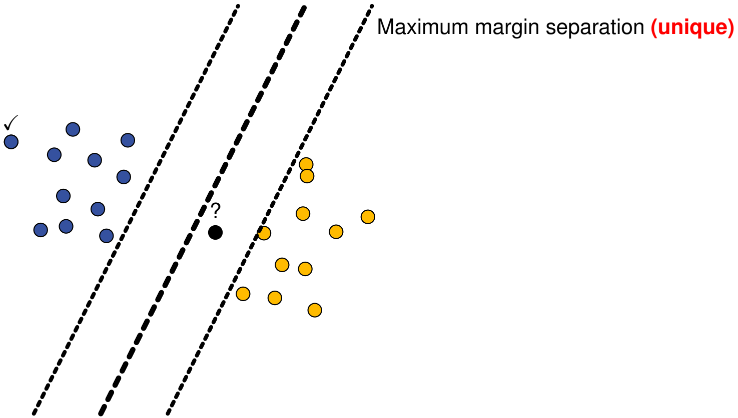

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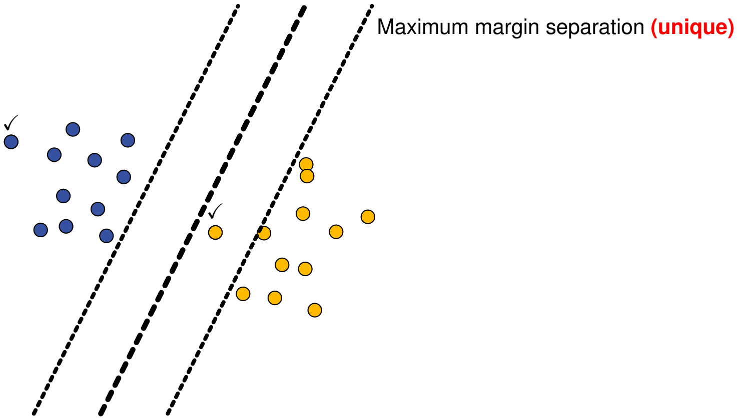

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This Work: Maximum margins in finite closure systems

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Adaptions:

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$\mathbb{R}^d \quad \rightarrow$

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Adaptions:

$\mathbb{R}^d \quad \rightarrow \quad E$ (*finite* ground set)

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$\mathbb{R}^d$	$\rightarrow$	E (*finite* ground set)
convexity	$\rightarrow$	

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all convex hulls	$\rightarrow$	

Definition:

closure operator c over E:

- $A \subseteq c(A)$
- $A \subseteq B \Rightarrow c(A) \subseteq c(B)$
- $c(c(A)) = c(A)$

Definition:

closure system $(E, \mathcal{C})$:

- $\emptyset, E \in \mathcal{C}$
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- **Which properties should be preserved?**
- **Goal: Efficient algorithm!**

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Maximum Margins in Finite Closure Systems

How to define margins in finite closure systems?

Monotone Linkage Functions (Mullat (1976))

Function $l : 2^E \to E$ such that

$$X \subseteq Y \Rightarrow l(X, e) \geq l(Y, e)$$

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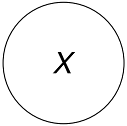

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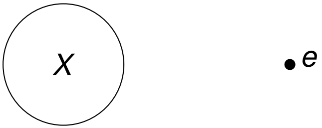

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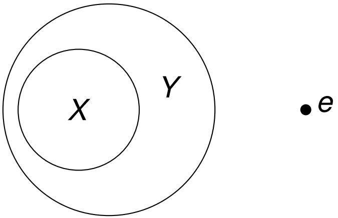

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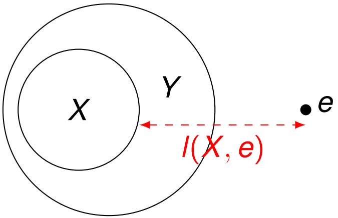

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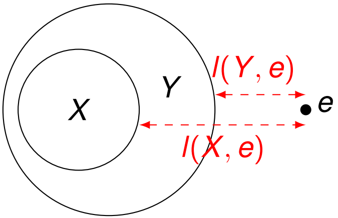

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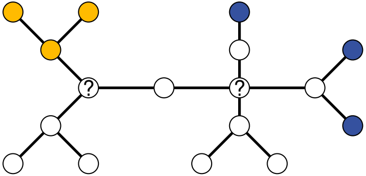
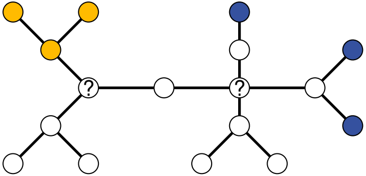


Trees



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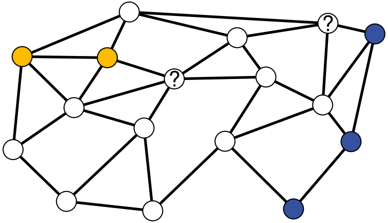

Graphs (e.g. molecule, social graph, . . .)

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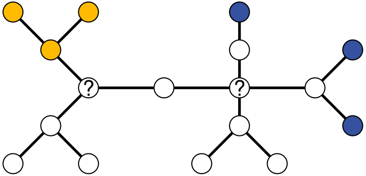

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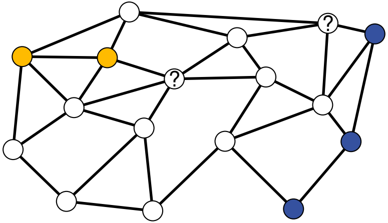

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- requires **semantically** meaningful definition of closure system/operator and linkage function

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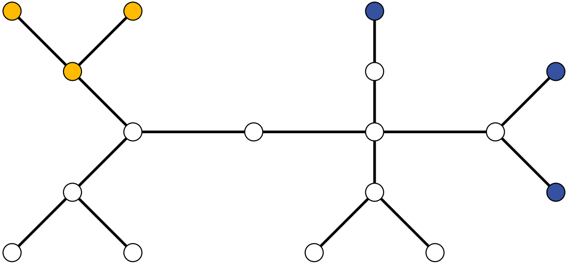

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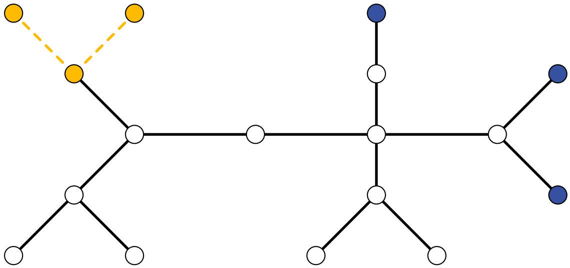

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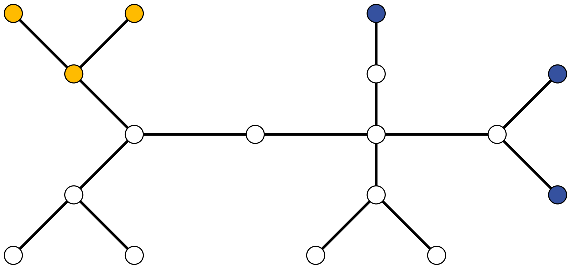

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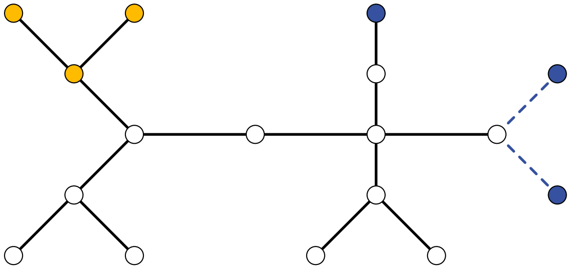

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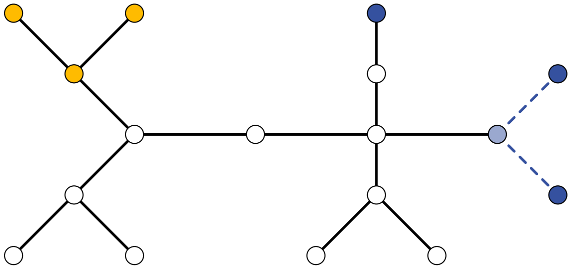

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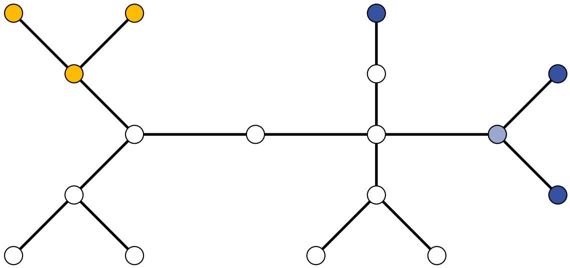

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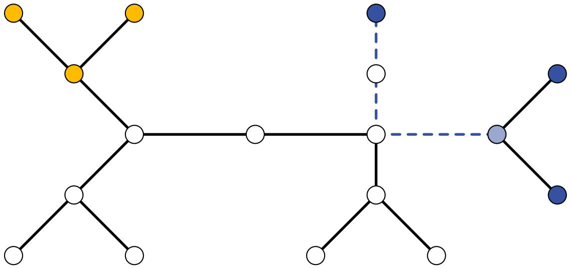

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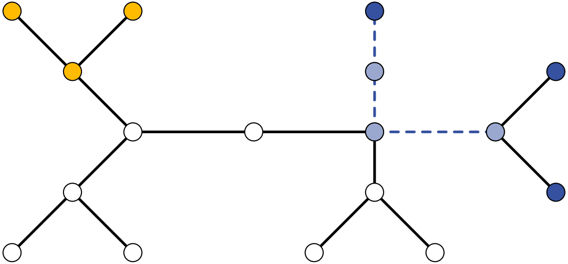

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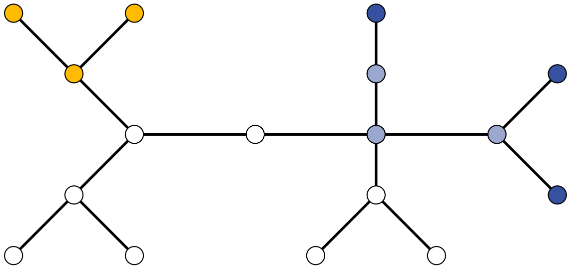

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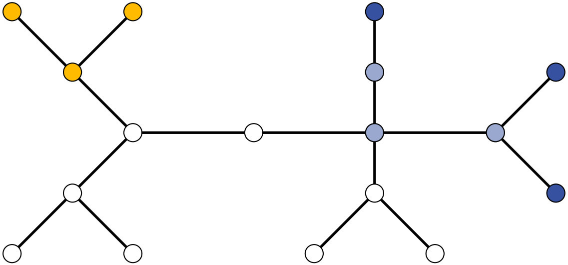

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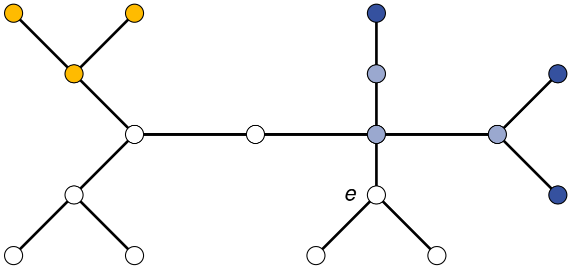

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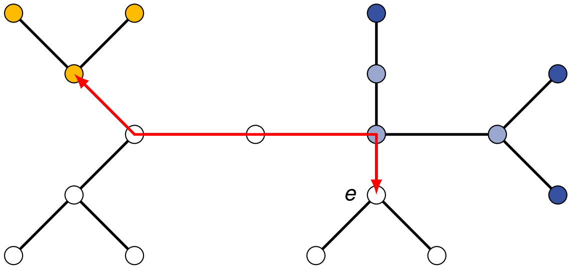

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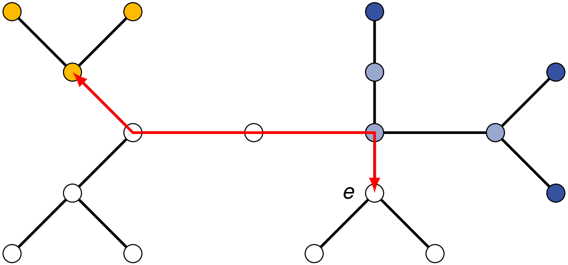

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Linkage from X to Y:

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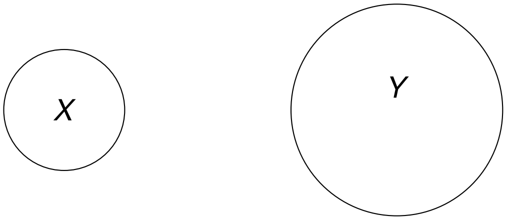

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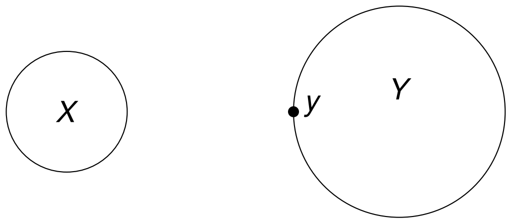

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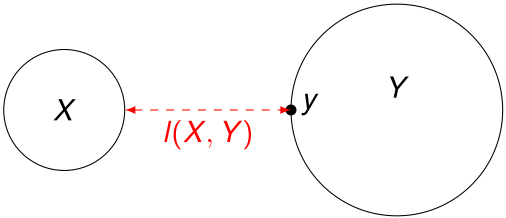

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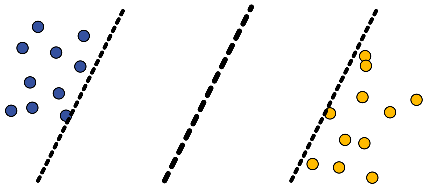
SVM:



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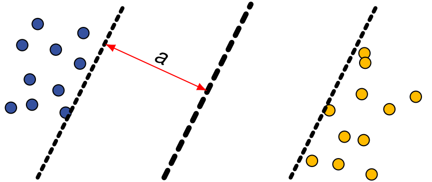

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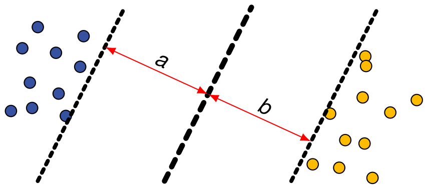

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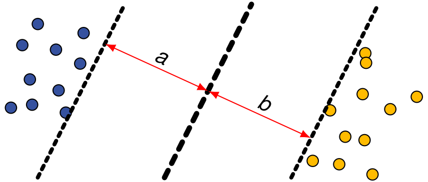

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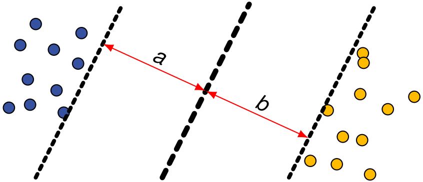

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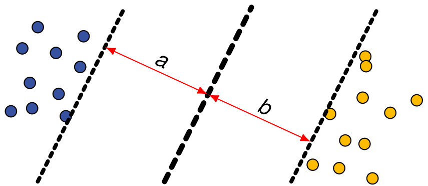

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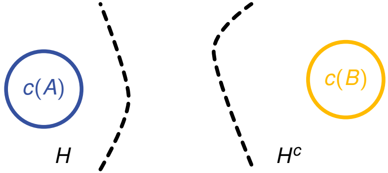

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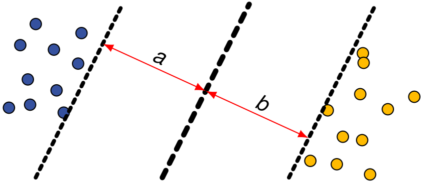

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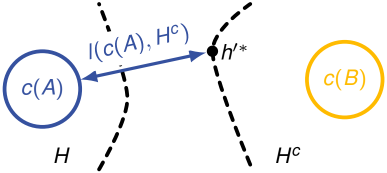

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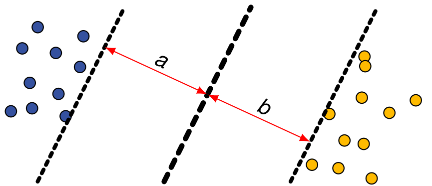

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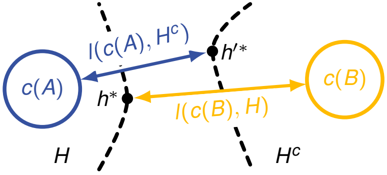

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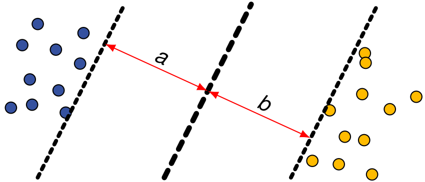

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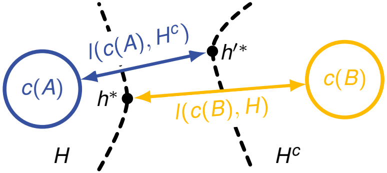

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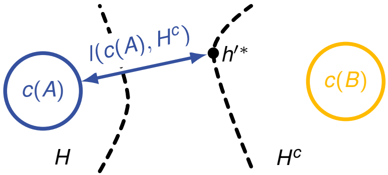

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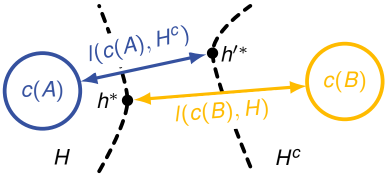

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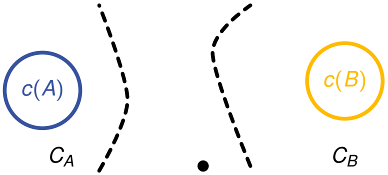

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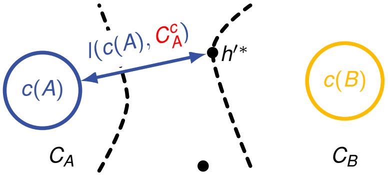

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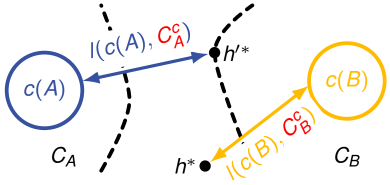

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Problem Definition

Maximum Margin Half-Space Separation Problem:

Given a monotone linkage closure system $(E, \mathcal{C}, l)$ and two sets $A, B \subseteq E$, **find** a half-space H which **maximizes** the margin, i.e.

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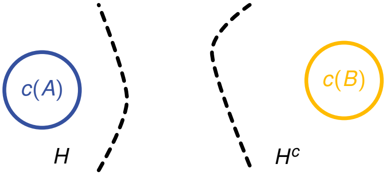

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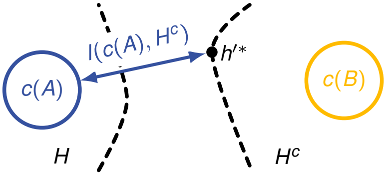

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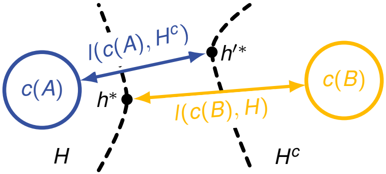

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All disjoint closed sets are half-space separable.

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Solutions:

1. restrict to Kakutani closure systems (e.g. trees, distributive lattices)
2. relax the problem to maximal disjoint closed sets

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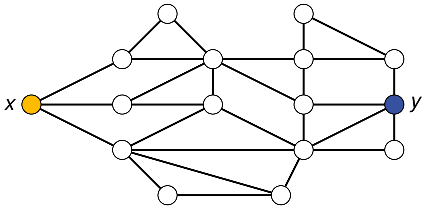

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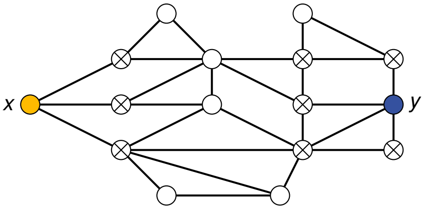

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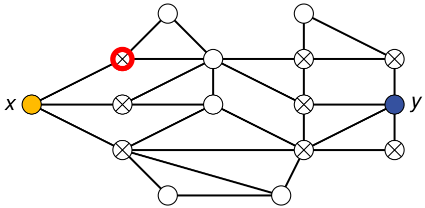

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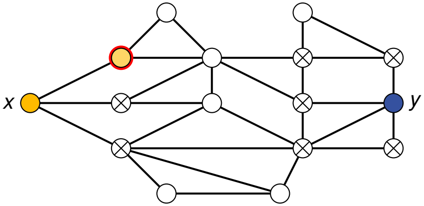

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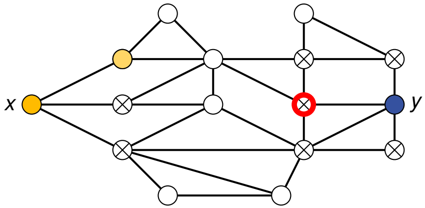

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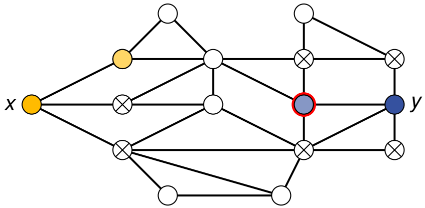

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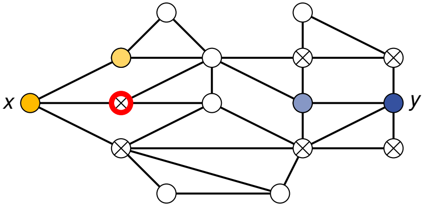

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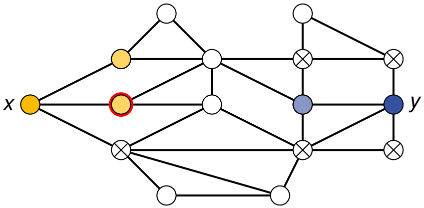

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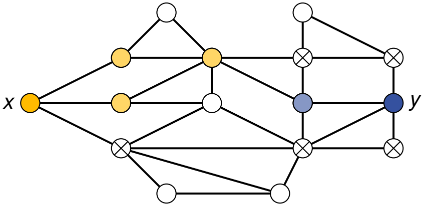

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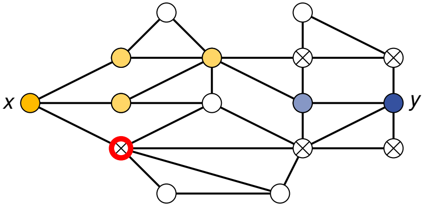

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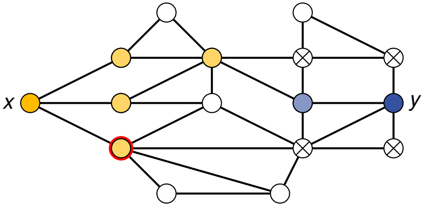

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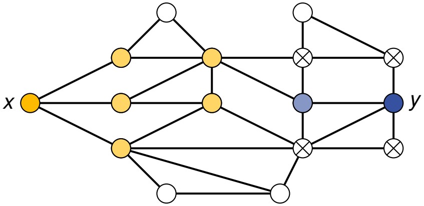

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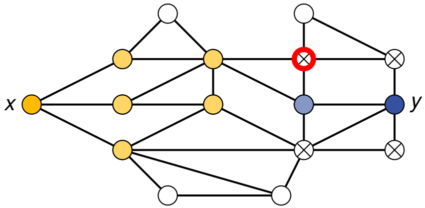

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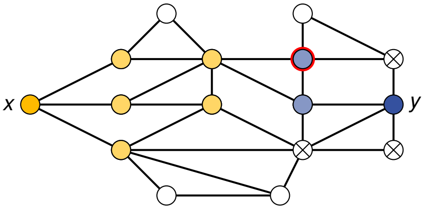

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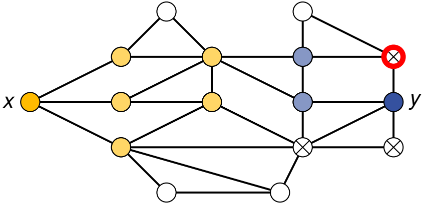

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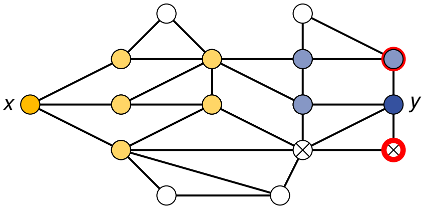

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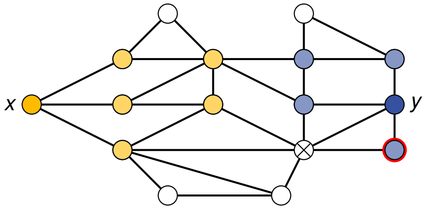

Maximum Margin Disjoint Closed Sets

Maximum-Margin Closed Set Separation Problem:

Given non-empty subsets A, B of E, *find* disjoint closed sets $C_A, C_B \in \mathcal{C}_c$ with $A \subseteq C_A, B \subseteq C_B$ that **maximize** the margin if $c(A) \cap c(B) = \emptyset$; o/w return "No".

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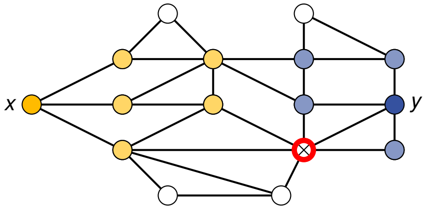

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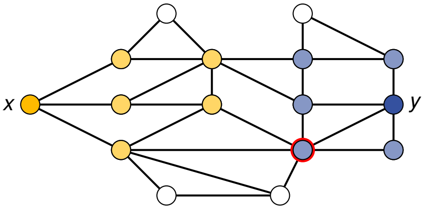

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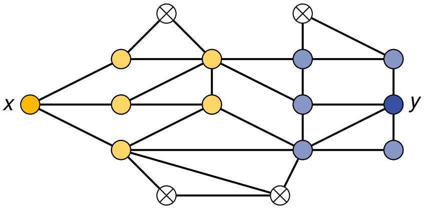

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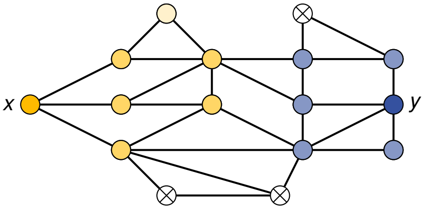

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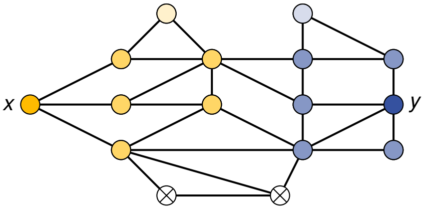

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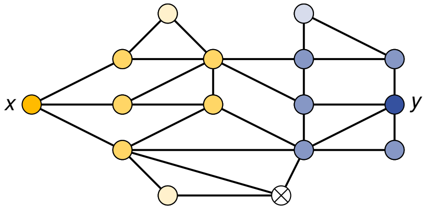

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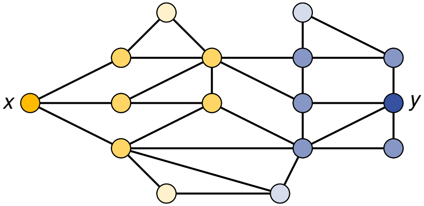

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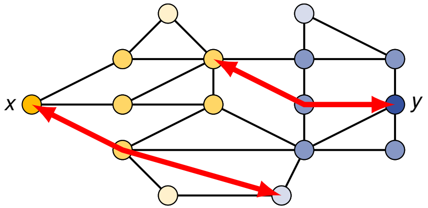

Results

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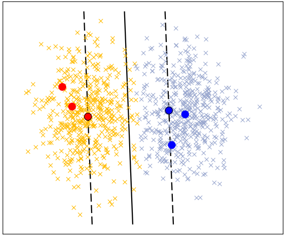
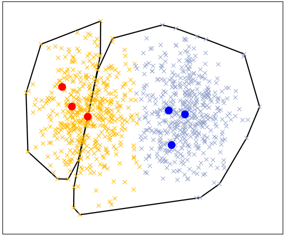
Corollary: The algorithm solves the Maximum Margin Half-Space Separation Problem if the closure system is Kakutani.

Experiments (Finite Point Sets)



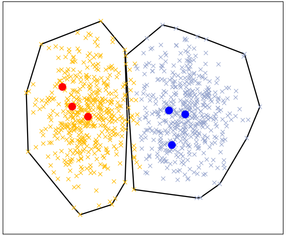
Support Vector Machine

(Vapnik, 1992)



Greedy Algorithm

(ECML/PKDD 2019)



Maximum Margin Algorithm
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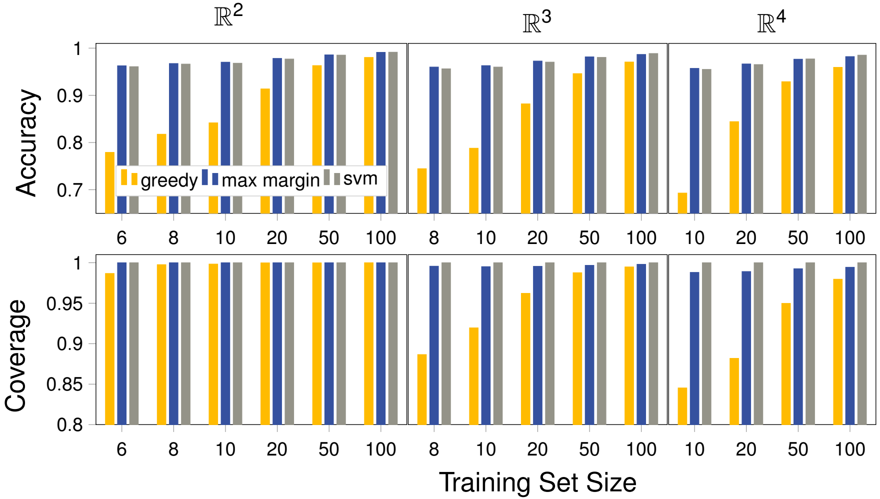
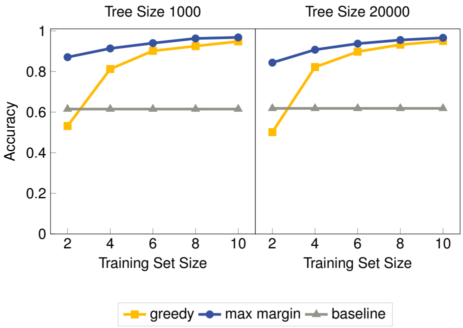
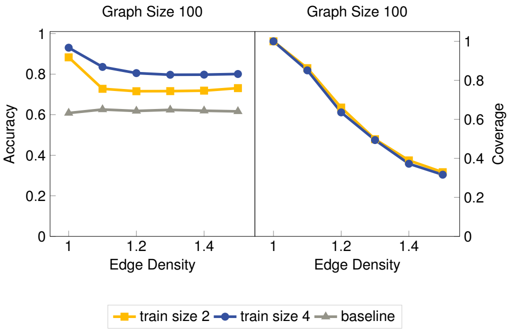


svm: (Vapnik, 1992), greedy: (ECML/PDKK 2019), max margin: (this work)

Experiments (Trees and Graphs)





Maximum Half-Space separation on trees

Maximum closed set separation on small graphs

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- allow approximate/fuzzy solutions

Further Details

Come to the Question and Answer Session or contact me:

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University of Bonn
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